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9. (a) $\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + 2 \cdot 2 + 4 \cdot 3 \\ -2 \cdot 2 + 3 \cdot 2 + 1 \cdot 3 \\ -4 \cdot 2 + 1 \cdot 2 + 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 \\ 1 \cdot 1 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 5 \end{pmatrix}$

10. (a) $A\vec{x} = 2 \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 0 \end{pmatrix}$

   (b) $A\vec{x} = 1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 5 \end{pmatrix}$

15. (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16. (a) $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   (b) $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

17. $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

18. $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. $E \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$, and $E^{-1} \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

20. $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $P_1 \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, and $P_2 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

33. $\vec{w} = 5\vec{u} + 7\vec{w}$. Then $A\vec{w} = A(5\vec{u} + 7\vec{v}) = A(5\vec{u}) + A(7\vec{v}) = 5A\vec{u} + 7A\vec{v}$. Note for later in the semester: this isn't that interesting to note for the standard basis as given here, but it's interesting that given the action of a matrix on *any* set of vectors, we can determine its action on any vector spanned by them.

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1. $l_{21} = 5$

$$\begin{aligned} 2x \quad &+3y &= 1 \\ &-6y &= 6 \end{aligned}$$

2. $y = -1$, so $2x - 3 = 1$. Therefore $x = 2$. Note that $2 \times (2, 10) - (3, 9) = (1, 11)$. If the right hand side changes to $(4, 44)$, $x = 8$, and $y = -4$.

4. $l = \frac{c}{a}$. The second pivot is $d - \frac{bc}{a}$, so we get:

$$\begin{aligned} ax \quad &+by &= f \\ &+\left(d - \tfrac{bc}{a}\right) y &= g - \frac{cf}{a} \end{aligned}$$

So $y = \frac{ag-bf}{ad-bc}$.

5. If (e.g.) the second RHS is 2, there is no solution (any number other than 20 will do the trick). If the second RHS is 20, there are infinitely many solutions. For example, $x = 0$, $y = 5$, and $y = 0$, $x = \frac{10}{3}$.

7. If $a = 0$, elimination breaks down temporarily, so swap the two equations:

$$\begin{aligned} 4x \quad &+6y &= 6 \\ &+3y &= -3 \end{aligned}$$

So $y = -1$, which gives $4x - 6 = 6$, or $x = 3$.

If $a = 2$, we get a permanent breakdown:

$$\begin{aligned} 2x \quad &+3y &= -3 \\ 4x \quad &+6y &= 6 \end{aligned}$$

$$\begin{aligned} 2x \quad &+3y &= -3 \\ &+0 &= 12 \end{aligned}$$

13.

$$\begin{aligned} 2x - 3y \quad &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned}$$

$$\begin{aligned} 2x - 3y \quad &= 3 \\ + 1y + z &= 1 \\ + 2y - 3z &= 2 \end{aligned}$$

$$\begin{aligned} 2x - 3y \quad &= 3 \\ + 1y + z &= 1 \\ -5z &= 0 \end{aligned}$$

So $z = 0$, giving $y + 0 = 1$, or $y = 1$, giving $2x - 3 = 3$, or $x = 3$.

17. Equal rows:

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned}$$

$$\begin{aligned} 2x - y + z &= 0 \\ 0 + 0 + 0 &= 0 \\ + 3y - z &= 2 \end{aligned}$$

$$\begin{aligned} 2x - y + z &= 0 \\ + 3y - z &= 2 \\ 0 + 0 + 0 &= 0 \end{aligned}$$

Nothing more to do!

Equal columns:

$$\begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2 \end{aligned}$$

$$\begin{aligned} 2x + 2y \quad + z &= 0 \\ + \quad \text{-1}z &= 0 \\ + \quad -2z &= 2 \end{aligned}$$

$$\begin{aligned} 2x + 2y \quad + z &= 0 \\ + \quad \text{-1}z &= 0 \\ + \quad 0 &= 2 \end{aligned}$$

Third pivot is missing!

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1. $BA = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$; $AB = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$; $ABD = \begin{pmatrix} 15 \\ 15 \\ 15 \end{pmatrix}$; DC is not allowed, nor is $A(B+C)$.

2. (a) The matrix A times the second column of B.
   (b) The first row of A times the matrix B.
   (c) The third row of A dot product with the fifth row of B.
   (d) This is either:
      - The first row of C dot the first column of DE, which is the first row of C dot the matrix D multiplied by the first column of E; or
      - The first row of CD dot the first column of E, which is the first row of C multiplied by the matrix D, dot the first column of E.

5. If $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix}$, and $A^3 = \begin{pmatrix} 1 & 3b \\ 0 & 1 \end{pmatrix}$. Also $A^5 = \begin{pmatrix} 1 & 5b \\ 0 & 1 \end{pmatrix}$, and $A^n = \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix}$.

If $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix}$, and $A^3 = \begin{pmatrix} 8 & 8 \\ 0 & 0 \end{pmatrix}$. Also $A^5 = \begin{pmatrix} 32 & 32 \\ 0 & 0 \end{pmatrix}$, and $A^n = \begin{pmatrix} 2^n & 2^n \\ 0 & 0 \end{pmatrix}$.

6. $(A+B)^2 = \begin{pmatrix} 10 & 4 \\ 6 & 6 \end{pmatrix}$; $A^2 + 2AB + B^2 = \begin{pmatrix} 16 & 2 \\ 3 & 0 \end{pmatrix}$. $(A+B)(A+B) = A^2 + AB + BA + B^2$.

26. $AB = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{pmatrix}$

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4. If (e.g.) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = 0$. However, since the (i,j) entry of A^TA is the dot product of the i^{th} column of A with the j^{th} column of A, the only way all entries could be 0 is if all the columns have length 0. Otherwise there will be at least one positive entry on the diagonal (since the diagonal entries are the squares of the lengths of the corresponding columns.)

Additional Problem

1. $A \cap B$ is a line.

2. Any linear combination of the two equations will do. For example, just adding the two gives C as the plane $2x - y = 1$.

3. Adding the two planes, but changing the constant will do the trick. So, for example, taking D to be the plane $2x - y = 10$ has $A \cap B \cap D$ empty, but $A \cap B$ and $B \cap D$ are both lines. The three lines are parallel, so there is no intersection.