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1. (a) $\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

3. $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{pmatrix}$.

6. Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$, and let $\vec{x} = [1,2,3]^T$. Then $A\vec{x} = [16,16,16]^T$. Elimination produces two rows of zeros, so there is only one pivot.

7. (a) To invert that step you should <u>add</u> 7 times row __1__ to row __3__.

   (b) $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$

   (c) $EE^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

14. $E_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $E_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $E_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{pmatrix}$

17. The equation is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix}$. Augmenting and row reducing, we get $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$. By back sub, this gives use $c = 1$, $b = 1$, $a = 2$.

24. $[A|\vec{u}] = \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 17 \end{array}\right)$ becomes $[U|\vec{c}] = \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right)$. So $\vec{x} = [5,-3]^T$.

25. $[A|\vec{b}] = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{array}\right)$ reduces to $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right)$. This gives $0 = 3$ as the last equation, so no solution. Changing the 6 to a 3 gives a solution (actually, an infinite number of them – see later).

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32. $AX = I$. x_1, x_2, and x_3 are exactly what you get on the right by augmenting the identity to A and doing Gauss-Jordan.

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7. (a) eqn 1 + eqn 2 must have right hand side 0, but since row 1 + row 2 = row 3, we get two equations with the same coefficients, but different outputs. Hence no solution.

   (b) There is a solution exactly if $b_1 + b_2 = b_3$.

   (c) It becomes $0 = b_3$.

22. $(A|I) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right) \xrightarrow{r_2 - 2r_1 \to r_2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \xrightarrow{r_1 - 3r_2 \to r_1} \left(\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right) = (I|A^{-1})$

26. $\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \xrightarrow{\frac{r_2}{2} \to r_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{r_1 - 2r_2 \to r_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So $E_{21} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, $D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, and $E_{12} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$. So $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{pmatrix}$.

27. $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 - 2r_1 \to r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 - 3r_3 \to r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{r_3 - r_2 \to r_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[r_1 - r_3 \to r_2]{r_2 - r_3 \to r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \xrightarrow{r_1 - r_2 \to r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

30. $A = \begin{pmatrix} a & b & b \\ a & a & b \\ a & a & a \end{pmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{pmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{pmatrix} \xrightarrow{r_3 - r_2 \to r_3} \begin{pmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{pmatrix}$. So if $a \neq 0$ and $a \neq b$, we have non-zero pivots a, $a-b$, and $a-b$. So the matrix is invertible.

    If $c = 0$, we have a row of all zeros, so C is not invertible. If $c = 2$, the first two rows are the same, so C is not invertible. If $c = 7$, the second and third columns are the same, so C is also not invertible (at this stage, students might have to argue that if $c = 7$, the transpose of A isn't invertible, so neither is A).

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5. $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix} \xrightarrow{r_3 - 3r_1 \to r_3} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$. So $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.

6. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{pmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{pmatrix} \xrightarrow{r_3 - 2r_2 \to r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{pmatrix} = U$. $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. So $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

7. $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[r3 - 3r_1 \to r_3]{r2 - 2r_1 \to r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{r_3 - 2r_2 \to r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = U$. So $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$, $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. So $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

8. (a) $E_{32}E_{31}E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac - b & -c & 1 \end{pmatrix} = E$

(b) $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} = L = A$.

15. $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$. We get $c_1 = 2$ and $4c_1 + c_2 = 11$, so $8 + c_2 = 11$, or $c_2 = 3$. Then $\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. We get $x_2 = 3$ and $2x_1 + 4x_2 = 2$, or $2x_1 + 12 = 2$, so $x_1 = -5$.

$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 8 & 17 \end{pmatrix}$.

So consider the augmented matrix $\left(\begin{array}{cc|c} 2 & 4 & 2 \\ 8 & 17 & 11 \end{array}\right) \xrightarrow{r_2 - 4r_1 \to r_2} \left(\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & 3 \end{array}\right)$. This gives $x_2 = 3$ and $2x_1 + 4x_2 = 2$, or $2x_1 + 12 = 2$, so $x_1 = -5$, as before.

16. $\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix} = \begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$. So $c_1 = 4$. We get $c_1 + c_2 = 5$, or $4 + c_2 = 5$, so $c_2 = 1$; and $c_1 + c_2 + c_3 = 6$, so $4 + 1 + c_3 = 6$, or $c_3 = 1$.

$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}4\\ 1\\ 1\end{pmatrix}$. So $x_3 = 1$. We get $x_2 + x_3 = 1$, or $x_2 + 1 = 1$, so $x_2 = 0$; and $x_1 + x_2 + x_3 = 4$, or $x_1 + 0 + 1 = 4$, so $x_1 = 3$.

$$A = \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{pmatrix}$$