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30. (a) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

    (b) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

    (c) $EM = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, $FEM = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $EFEM = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $EEFEM = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $FEEFEM = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

    (d) $M^{-1} = FEEFE$, so $M = E^{-1}F^{-1}E^{-1}E^{-1}F^{-1} = ABAAB$.

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10. $A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/4 & 0 & 0 \\ 1/5 & 0 & 0 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{pmatrix}$

11. (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

    (b) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

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8. You have n choices for where the 1 in the first row goes. That leaves $(n-1)$ choices for where the 1 in the second row goes, then $(n-2)$ choices for where the 1 goes in the third row, and so on. Altogether, there are $n!$ possibilities.

13. (a) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

    (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

22. First matrix: $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{3}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

    Second matrix: $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 4 & 1 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$.

39. (a) All diagonal entries are 1, so for each i, $q_i^T q_i = 1$, but $||q_i||^2 = q_i^T q_i$.
    (b) All non-diagonal entries are 0, so for each $i \neq j$, $q_i^T q_j = 0$.
    (c) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. (There are three other possibilities)

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9. (a) $\begin{pmatrix} x \\ y \end{pmatrix}$ with x and y integers. (Note: this is a *lattice*, an important structure in other areas of math.)
   (b) Two lines intersecting at the origin.

10. (a) Yes;
    (b) No (doesn't contain the zero vector);
    (c) No (not closed under addition);
    (d) Yes;
    (e) Yes;
    (f) No (Not closed under scalar multiplication, e.g. by a negative scalar).

12. Many possibilities, like $v_1 = [0, 0, -2]^T$, and $v_2 = [4, 0, 0]^T$.

13. P_0 is given by $x + y - 2z = 0$. Many answers. E.g. $v_1 = [1, 1, 1]$ and $v_2 = [2, 0, 1]$. Then $v_1 + v_2 = [3, 1, 2]$, and $3 + 1 - 2 \times 2 = 0$, as required.

16. Suppose P is a plane through $(0, 0, 0)$ and L is a line through $(0, 0, 0)$. The smallest vector space containing both P and L is either a point or a line (that is, it's either the zero vector space, or L itself).

20. (a) Only for multiples of $[1, 2, -1]^T$.

    (b) Any vector with $b_1 + b_3 = 0$.

22. First system: all vectors in $\mathbb{R}^3$; Second system: all vectors for which $b_3 = 0$; Third system: all vectors for which $b_2 = b_3$.

23. If we add an extra column $\vec{b}$ to a matrix A, then the column space gets larger unless $\vec{b} \in C(A)$.
    For example, if we add the column $[1, 1, 0]^T$ to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, the column space doesn't get larger, but if we add the column $[0, 0, 1]^T$, it does. $A\vec{x} = \vec{c}$ is solvable exactly if the column space doesn't get larger because in that case, $\vec{c} \in C(A)$, which is exactly the condition necessary for the equation to have a solution.

24. For two square matrices, any non-singular matrix A and singular matrix B will do. Specifically, if B is the zero matrix, we're done.

27. (a) False. This set doesn't contain the zero vector, so can't be a subspace.
    (b) True.
    (c) True.
    (d) False. For example, if $A = I$, then $C(A) = \mathbb{R}^n$, but $C(A - I) = \{\vec{0}\}$.

28. Many examples. Easiest for the first part: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. For the second part, any rank 1 matrix will do. For example, a matrix all of whose columns are the same and are not all zeros.

32. $C(AB) \subseteq C(A)$, so by adding the columns of AB to the matrix A (to get $[AAB]$), we don't expand the column space. If (e.g.) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = 0$, so $C(A^2)$ is smaller than $C(A)$. An n by n matrix has $C(A) = \mathbb{R}^n$ exactly when A is an invertible matrix.

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1. (a) $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (also accept the row echelon form, $\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$). Free variables are x_2, x_4, and x_5.
   (b) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (also accept the row echelon form, $\begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$). Only x_3 is free.

12. $A = \begin{pmatrix} 1 & -3 & -1 \end{pmatrix}$. The variables y and z are free. The special solutions are $\underline{[3, 1, 0]^T}$ and $\underline{[1, 0, 1]^T}$.

1. (a) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
   (b) $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
   (c) $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

31. Accept either RREF or REF:

(a) $\begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, or same with all 1's in the first row.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (RREF is the same.)

Extra Problem

A quadratic is $y = ax^2 + bx + c$. The system of equations is therefore:

$$\begin{aligned} -7 &= a + b + c \\ -16 &= 4a + 2b + c \\ -33 &= 9a + 3b + c \end{aligned}$$

Or

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -7 \\ -16 \\ -33 \end{pmatrix}.$$

Gaussian elimination takes the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -7 \\ 4 & 2 & 1 & -16 \\ 9 & 3 & 1 & -33 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & -7 \\ 0 & -2 & -3 & 12 \\ 0 & 0 & 1 & -6 \end{array}\right)$$

So by back-sub, we get $c = -6$, $b = 3$, $a = -4$. This can be checked by plugging in the x values.