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4. $RREF(A) = \begin{pmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix}$, so the special solutions are $[3, 1, 0]^T$ and $[5, 0, 1]$. $RREF(B) = \begin{pmatrix} 1 & -3 & -5 \\ 0 & 0 & 1 \end{pmatrix}$, so the special solution is $[0, 1, 0]^T$. For an n by m matrix the number of pivot variables plus the number of free variables is $\underline{\ n\ }$.

5. (a) False. The square zero matrix, for example, has all variables free.
   (b) True. If A is invertible, then the unique solution to $A\vec{x} = \vec{c}$ is $\vec{x} = A^{-1}\vec{c}$. Equivalently, the RREF of an invertible matrix is the identity. Equivalently, all columns (or rows) of an invertible matrix are lin. ind.
   (c) True. The number of columns is the number of variables. So certainly, there are no more free variables than columns.
   (d) True. There can be no more than one pivot variable per row.

14. Column 5 has no pivot. $N(A)$ is one-dimensional, with basis $[1, 0, 1, 0, 1]^T$, since $1 \cdot c_1 + 0 \cdot c_2 + 1 \cdot c_3 + 0 \cdot c_4 + 1 \cdot c_5 = \vec{0}$. Such a matrix might be $\begin{pmatrix} \textcircled{1} & 0 & 0 & 0 & -1 \\ 0 & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{pmatrix}$.

17. $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$. Note that $c_1 + c_2 + 2c_3 = \vec{0}$, so the required vector is in the nullspace, and that the pivot columns (1 and 2) are a basis for the column space.

20. There are many. Anything whose RREF is $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ will do the trick.

21. Impossible, since rank + nullity = 3, so the column space and row space cannot have the same dimension.

22. If $AB = 0$, then the column space of B is contained in the <u>nullspace</u> of A. If $\vec{c} \in C(B)$, then $\vec{c} = B\vec{x}$ for some vector $\vec{x}$. Since $AB\vec{x} = 0\vec{x} = \vec{0}$, and by associativity, we get $\vec{0} = A(B\vec{x}) = A\vec{c}$. So $\vec{c} \in N(A)$.

33. (c) is the only correct definition of rank.

41. $\begin{pmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$, and $\begin{pmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & 2 \end{pmatrix}$

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4. The RREF of the matrix is $\begin{pmatrix} \textcircled{1} & 3 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So columns 1 and 3 form a basis for $C(A)$.

   Note that $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, so the particular solution is $x_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$. By backsub, we find that the basis for $N(A)$ is $\left(\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right)$. Therefore the complete solution is

$$\vec{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

5. The RREF of $\left(\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right)$ is $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right)$. So the system has a solution if and only if $b_3 - b_2 - 2b_1 = 0$. If that's the case, then $\vec{x}_p = \begin{pmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{pmatrix}$, and the special solution is $[2, 0, 1]^T$. So the general solution is $\vec{x} = \begin{pmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

10. Easiest solution is $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, but there are many.

16. The largest possible rank of a 3 by 5 matrix is <u>3</u>. Then there is a pivot in every <u>row</u> of U and R. The solution to $A\vec{x} = \vec{b}$ *always exists*. The column space of A is $\mathbb{R}^3$. An example is $A = \begin{pmatrix} \textcircled{1} & 0 & 0 & 1 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 1 \end{pmatrix}$.

22. If $A\vec{x} = \vec{b}$ has infinitely many solutions, then $N(A)$ has non-zero dimension. So if one solution to $A\vec{x} = \vec{B}$ exists (i.e. $\vec{B} \in C(A)$), you can add any vector in the nullspace to get another solution. If $\vec{B} \notin C(A)$, then there is no solution.

33. $A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$.