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24. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, or any matrix with full column rank, but not full row rank.

    (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, or any matrix with full row rank, but not full column rank.

    (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, or any matrix with neither full row rank, nor free column rank.

    (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, or any invertible matrix

25. (a) $r < m$

    (b) $r = m < n$

    (c) $r = n < m$

    (d) $r = m = n$

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2. Taking each vector to be the column of a matrix, then finding its RREF gives the first three columns as pivot columns. Hence a maximal linearly independent set is the first three vectors.

7. The v_i's are linearly dep. if there is a set of non-zero c_1, c_2, c_3 that solve $c_1v_1+c_2v_2+c_3v_3 = 0$. By plugging in the expressions for each v_i in terms of the w_i's, we get

$$(c_2 + c_3)w_1 + (c_1 - c_3)w_2 - (c_1 + c_2)w_3 = 0.$$

One set of solutions is $c_1 = c_3 = 1$, $c_2 = -1$. $[v_1v_2v_3] = [w_1w_2w_3]A$, where A is the singular matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

8. We want to show that the v_i's are independent. So write

$$\begin{aligned} \vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &\Leftrightarrow \vec{0} = c_1\vec{w}_2 + c_1\vec{w}_3 + c_2\vec{w}_1 + c_2\vec{w}_3 + c_3\vec{w}_1 + c_3\vec{w}_2 \\ &\Leftrightarrow \vec{0} = (c_2 + c_3)\vec{w}_1 + (c_1 + c_3)\vec{w}_2 + (c_1 + c_2)\vec{w}_3 \end{aligned}$$

Since the $\vec{w}_i$'s are independent, this means that

$$c_2 + c_3 = 0$$
$$c_1 + c_3 = 0$$
$$c_1 + c_2 = 0$$

or $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. By row-reducing, we see that the matrix has rank 3, so its nullspace is just the zero vector. So the only solution is $c_1 = c_2 = c_3 = 0$. Therefore the v_i's are independent.

12. The vector $\vec{b}$ is in the subspace spanned by the columns of A when $\underline{A\vec{x} = \vec{b}}$ has a solution. The vector $\vec{c}$ is in the row space of A when $\underline{A^T\vec{y} = \vec{c}}$ has a solution. It is *false* that if the zero vector is in the row space then the rows are dependent. In fact, the zero vector is always in the row space by definition of a space.

21. (a) The equation $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$ because A has full column rank, since its columns are independent, so $N(A) = \{\vec{0}\}$.
    (b) If $\vec{b}$ is in $\mathbb{R}^5$, then $A\vec{x} = \vec{b}$ is solvable because the basis vectors span $\mathbb{R}^5$.

45. Let $v_1, \ldots, v_\alpha$ be a basis for V, and let $w_1, \ldots, w_\beta$ be a basis for W. Then we know $\alpha + \beta > n$. Since any basis of $\mathbb{R}^n$ has n vectors in it, the set $\{v_1, \ldots, v_\alpha, w_1, \ldots, w_\beta\}$ must be linearly dependent. Therefore there exists a set of scalars $c_1, \ldots, c_\alpha, d_1, \ldots, d_\beta$, not all zeros such that

$$c_1v_1 + \ldots + c_\alpha v_\alpha + d_1w_1 + \ldots d_\beta w_\beta = \vec{0}.$$

Therefore $c_1v_1 + \ldots + c_\alpha v_\alpha = -(d_1w_1 + \ldots d_\beta w_\beta)$ is a non-zero vector in $V \cap W$.

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3.
   - Basis for $C(A)$: $[1, 1, 0]$, $[3, 4, 1]$.
   - Basis for $C(A^T)$: $[0, 1, 2, 3, 4]$, $[0, 1, 2, 4, 6]$. (Or $[0, 1, 2, 3, 4]$, $[0, 0, 0, 1, 2]$.)
   - Basis for $N(A)$: $[1, 0, 0, 0, 0]$, $[0, -2, 1, 0, 0]$, $[0, 2, 0, -2, 1]$
   - Basis for $N(A^T)$ (after finding $RREF(A^T)$): $[1, -1, 1]$.

4. When they exist, these are not usually unique.
   (a) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.
   (b) Impossible. This must be a 3×3 matrix, which means $dim(C(A)) + dim(N(A)) = 3$.
   (c) $\begin{pmatrix} 1 & 0 \end{pmatrix}$ (I think this really must be the simplest possible example).
   (d) $\begin{pmatrix} 9 & 3 \\ 3 & -1 \end{pmatrix}$

(e) Impossible. Since $C(A)^\perp = N(A^T)$ and $C(A^T)^\perp = N(A)$, then if $C(A) = C(A^T)$, then $N(A) = N(A^T)$.

11. (a) $r \leq m$ (always true), $r < n$ (since $r = dim(C(A)) < n$, since it is possible that there are no solutions).

    (b) The condition is equivalent to $dim(N(A^T)) > 0$. Since $dim(N(A^T)) + dim(C(A)) = n$, and $dim(C(A)) < n$, this must be true.

18. The combination $r_3 - 2r_2 + r_1 = \vec{0}$. So multiples of $[1, -2, 1]$ are in $N(A^T)$. Similarly, since $c_3 - 2c_2 + c_1 = \vec{0}$, multiples of $[1, -2, 1]$ are also in $N(A)$.

19. (a) The row echelon form of $\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{array}\right)$ is $\left(\begin{array}{ccc} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - b_2 - b_1 \end{array}\right)$. So $N(A^T)$ has basis $[-1, -1, 1]^T$.

    (b) The row echelon form of $\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{array}\right)$ is $\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 2b_2 + 2b_1 \\ 0 & 0 & b_4 - 3b_2 + 4b_1 \end{array}\right)$. So a basis of $N(A^T)$ is $\{[2, -2, 1, 0], [4, -3, 0, 1]\}$.

24. $A^T\vec{y} = \vec{d}$ is solvable when $\vec{d} \in C(A^T)$, the row space. The solution is unique when the <u>left nullspace</u> contains only the zero vector.

25. (a) True. These are both the rank of the matrix.

    (b) False. Left nullspace of A^T is the nullspace of A, which lives in a different dimension than the left nullspace of A for any non-square matrix.

    (c) False. Row space and column space are the same for any invertible matrix.

    (d) True. This is trivially true if $A^T = A$, and multiplying the right side by -1 doesn't change the space spanned by the columns.

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9. If $A^TA\vec{x} = \vec{0}$, then $A\vec{x} = \vec{0}$. Reason: $A\vec{x}$ is in the nullspace of A^T and also in the <u>column space</u> of A and those spaces are <u>orthogonal complements</u>.

10(a) The column space of a matrix is perpendicular to its left nullspace. However, since the matrix is symmetric, the left nullspace is exactly the nullspace, so the column space is perpendicular to the nullspace.

14. The column spaces of A and B are planes in $\mathbb{R}^3$. $[AB] = \begin{pmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{pmatrix}$. The RREF of this is $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. So $A\begin{pmatrix} 3 \\ 1 \end{pmatrix} = B\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (the first column of B) is in both column spaces.

17. If S is the subspace of $\mathbb{R}^3$ containing only the zero vector, then $S^\perp = \mathbb{R}^3$. If S is spanned by $(1,1,1)$, then $S^\perp$ is the plane spanned by $(1,-1,0)$ and $(0,1,-1)$. If S is spanned by $(1,1,1)$ and $(1,1,-1)$, then $S^\perp$ is spanned by $(1,-1,0)$.

21. If S is spanned by $(1,2,2,3)$ and $(1,3,3,2)$, then $S^\perp$ is spanned by $(0,-1,1,0)$ and $(-5,1,0,1)$. This is the same as solving $A\vec{x} = \vec{0}$ for $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix}$ (i.e. finding the nullspace of this matrix.)

Extra Problem

1. $Rank(A) = 6$.

2.
   - Basis for $C(A)$: (column space):
     $(1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 5)$
     $(0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$
     $(0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -11)$
     $(0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; -6)$
     $(0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; -4)$
     $(0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -4)$
   - Basis for $N(A^T)$ (left nullspace):
     $\begin{pmatrix} 5 & 0 & -11 & -6 & -4 & -4 & -1 \end{pmatrix}$
   - Basis for $C(A^T)$ (row space):
     $(1 \;\; 0 \;\; 0 \;\; 0 \;\; -5 \;\; 0 \;\; 0 \;\; 2 \;\; -3 \;\; -2)$
     $(0 \;\; 1 \;\; 0 \;\; 0 \;\; 3 \;\; 0 \;\; 0 \;\; 0 \;\; 3 \;\; -3)$
     $(0 \;\; 0 \;\; 1 \;\; 0 \;\; -2 \;\; 0 \;\; 0 \;\; 2 \;\; 1 \;\; -2)$
     $(0 \;\; 0 \;\; 0 \;\; 1 \;\; -3 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; -2)$
     $(0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; -2 \;\; 1)$
     $(0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -2 \;\; 2 \;\; 1)$
   - Basis for $N(A)$ (nullspace):
     $(5 \;\; -3 \;\; 2 \;\; 3 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$
     $(-2 \;\; 0 \;\; 2 \;\; 0 \;\; 0 \;\; -1 \;\; 2 \;\; 1 \;\; 0 \;\; 0)$
     $(3 \;\; -3 \;\; -1 \;\; 1 \;\; 0 \;\; 2 \;\; -2 \;\; 0 \;\; 1 \;\; 0)$
     $(2 \;\; 3 \;\; 2 \;\; 2 \;\; 0 \;\; -1 \;\; -1 \;\; 0 \;\; 0 \;\; 1)$

3. $$RREF([A|c]) = \begin{pmatrix} 1 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & -3 & -2 & 9 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 2 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & -1 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
   By setting all the free variables to zero, we get $x_p = [9,1,5,6,0,0,-3,0,0,0]$.

4. x_p is a linear combination of basis vectors in $C(A^T)$ (the row space) and $N(A)$ (the nullspace).

5. In order to express x_p as a linear combination of basis vectors in $C(A^T)$ (the row space) and $N(A)$ (the nullspace), create a 10×11 matrix whose first six columns are the basis vectors for $C(A^T)$, the next four columns the basis vectors for $N(A)$, and the last column x_p. Then the last column of this matrix's RREF is the coefficients of x_p as a linear combination of these basis vectors. (Bonus points: The coefficient vector turns out to be $\frac{1}{193}\begin{pmatrix} 26620 & 13768 & -35461 & -11647 & -13815 & 2005 & -14067 & 35507 & 30730 & 12138 \end{pmatrix}$.)