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24. The first column of A^{-1} is orthogonal to the space spanned by row 2 through n of A.

28. (a) Two planes in $\mathbb{R}^3$ cannot be orthogonal. In this case, the vector $[1, -1, 0]^T$ and its multiples form the intersection of the two planes.

    (b) The orthogonal complement of a 2-D subspace of $\mathbb{R}^5$ is 3-D, but there are only two basis vectors given.

    (c) The subspaces spanned by $[1, 0]^T$ and $[1, 1]^T$ only intersect at the origin, but aren't orthogonal.

29. The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ has $\vec{v} = [1, 2, 3]^T$ in the column and row spaces. The matrix $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -4 \\ 3 & 1 & -5 \end{pmatrix}$ has $\vec{v}$ in the column and nullspace. But $\vec{v}$ cannot be both in the column space and the left nullspace, or in the row space and nullspace, since the intersections of both those pairs is $\vec{0}$.

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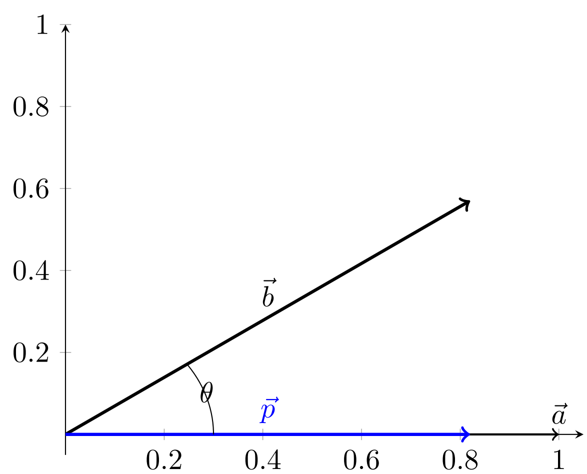
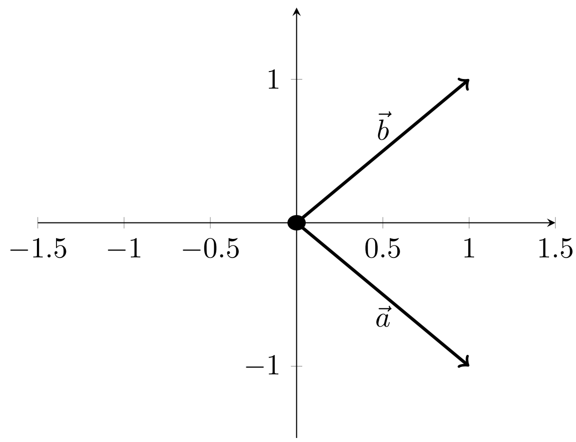
2. (a) $\hat{x} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \dfrac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \cos\theta$. So $\vec{p} = \hat{x}\vec{a} = \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ 0 \end{pmatrix}$.



   (b) $\hat{x} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \dfrac{\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = 0$. So $\vec{p} = 0\vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



5. $P_1 = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$, and $P_2 = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$. Note that $P_1P_2 = 0$. This makes sense, because the vectors are orthogonal, so projecting onto one, then the other always gives the zero vector.

6. $\vec{p_1} = P_1\vec{b} = \frac{1}{9}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$; $\vec{p_2} = P_2\vec{b} = \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$; $\vec{p_3} = P_3\vec{b} = \frac{1}{9}\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$. $p_1 + p_2 + p_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

7. $P_3 = \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$. The three do indeed add up to I.

11. (a) $\vec{p} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$; $\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

    (b) $\vec{p} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$; $\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

12. $P_1 = \frac{1}{9}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2 = \frac{1}{9}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $P_1\vec{b} = \frac{1}{9}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. P_2^2 is indeed the identity.

17. $(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$. When P projects on the column space of A, $I - P$ projects onto its left nullspace.

19. Two possible vectors are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$. Then $P = \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$.

20. The vector $\vec{e} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ (the vector of coefficients) is perpendicular to the plane. So $Q = \frac{\vec{e}\vec{e}^T}{\vec{e}^T\vec{e}} =$

$\frac{1}{6}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$. So $P = I - Q = \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$, as before.

24. The nullspace of A^T is orthogonal to the column space of $C(A)$. So if $A^T\vec{b} = \vec{0}$, the projection of $\vec{b}$ onto $C(A)$ should be $\vec{p} = \underline{\vec{0}}$. Indeed, $P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = A(A^TA)^{-1}(A^T\vec{b}) = A(A^TA)^{-1}\vec{0} = \vec{0}$.

26. If an $m \times m$ matrix has rank m, then it is invertible. So if $A^2 = A$, then $A^{-1}A^2 = A^{-1}A \Rightarrow A = I$.

31. You would take the dot product of $\vec{b} - \vec{p}$ with each of the $\vec{a}_i$'s. If all the dot products are zero, then $\vec{p}$ is the orthogonal projection of $\vec{b}$ onto the subspace spanned by the $\vec{a}_i$'s.

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28. If the columns of A are not independent, find the row echelon form of A to figure out the pivot columns. Remove all free columns from A to create a matrix B with the same column space, but with full row rank. Then $P = B(B^TB)^{-1}B^T$ is the projection onto $C(B) = C(A)$.

Extra Problem

1. Create a matrix B that is A augmented by $\vec{c}$. Its RREF is $\begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Since the last row is not all zeros, $\vec{c}$ cannot be written as a linear combination of the columns of A, so it's not in $C(A)$.

2. $A^TA = \begin{pmatrix} 14 & 5 & 13 & 1 \\ 5 & 5 & -5 & 10 \\ 13 & -5 & 41 & -28 \\ 1 & 10 & -28 & 29 \end{pmatrix}$, and $A^T\vec{c} = [22, 12, 8, 14]^T$.

3. We saw in part 1 of the question that A has rank 2, so $N(A)$ is two-dimensional. Since we know that $N(A^TA) = N(A)$, $N(A^TA) \neq \{\vec{0}\}$, so $A^TA\hat{x} = A^T\vec{c}$ does not have a unique solution.

4. Form the augmented matrix $[A^TA|A^T\vec{c}]$. Its RREF is $\begin{pmatrix} 1 & 0 & 2 & -1 & 10/9 \\ 0 & 1 & -3 & 3 & 58/45 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. So $\hat{x}_p = [10/9, 58/45, 0, 0]^T = \frac{1}{45}[50, 58, 0, 0]^T$. The special solutions of are $\hat{s}_1 = [-2, 3, 1, 0]^T$ and $\hat{s}_2 = [1, -3, 0, 1]^T$, so the general solution of the normal equation is

$$\hat{x} = \frac{1}{45}\begin{pmatrix} 50 \\ 58 \\ 0 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

5. $\vec{p} = A\hat{x} = \frac{1}{45}[-8, 100, 266]^T$, and $\vec{e} = \vec{c} - \vec{p} = \frac{1}{45}[8, -10, 4]^T$.

6. Since A doesn't have full column rank, we can't just plug it into the formula for projection matrices. However, we can if we drop the two free columns (the 3[rd] and 4[th]). If C is the matrix consisting of the two remaining pivot columns of A, then $C(A) = C(C)$, so $P = C(C^TC)^{-1}C^T = \frac{1}{45}\begin{pmatrix} 29 & 20 & -8 \\ 20 & 20 & 10 \\ -8 & 10 & 41 \end{pmatrix}$.

7. The projection matrix onto $C(A^T)$ is the matrix $E(E^TE)^{-1}E^T$, where E is the matrix formed by the pivot columns of A^T. By taking the RREF of A^T, we see that the first two columns are pivot columns. So $E = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 5 & 4 \\ -4 & -2 \end{pmatrix}$, giving $P = \begin{pmatrix} 19 & 9 & 11 & 8 \\ 9 & 6 & 0 & 9 \\ 11 & 0 & 22 & -11 \\ 8 & 9 & -11 & 19 \end{pmatrix}$. This gives

$$\hat{x}_{\text{row}} = P\hat{x}_p = \frac{1}{1485}\begin{pmatrix} 1472 \\ 798 \\ 550 \\ 922 \end{pmatrix}.$$

So we get

$$\hat{x} = \frac{1}{1485}\begin{pmatrix} 1472 \\ 798 \\ 550 \\ 922 \end{pmatrix} + c_1\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$