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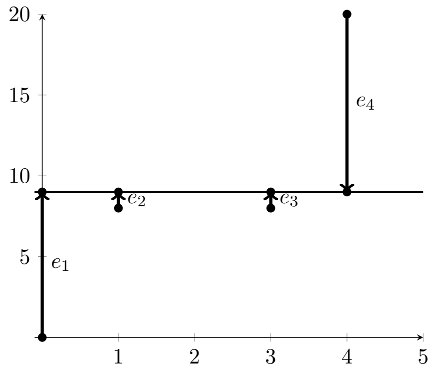
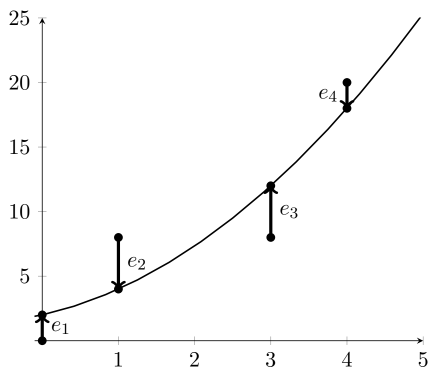
1. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$. $\vec{b} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$. So $A^T A\hat{x} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$ and $A^T\vec{b} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$. $\hat{x} = (A^T A)^{-1} A^T \vec{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. So $\vec{p} = A\hat{x} = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix}$, and $E = |\vec{e}|^2 = (\vec{b} - \vec{p})^2 = 44$.

5. $A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. So $A^T A = 4$, and $A^T\vec{b} = 36$. Therefore $\hat{x} = 9$. Note that this is the average of the y values.



9. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$. $\hat{x} = (A^T A)^{-1} A\vec{b} = \begin{pmatrix} 2 \\ \frac{4}{3} \\ \frac{2}{3} \end{pmatrix}$. Best fit parabola is $2 + \frac{4}{3}x + \frac{2}{3}x^2$, and $\vec{p} = \begin{pmatrix} 2 \\ 4 \\ 12 \\ 18 \end{pmatrix}$.

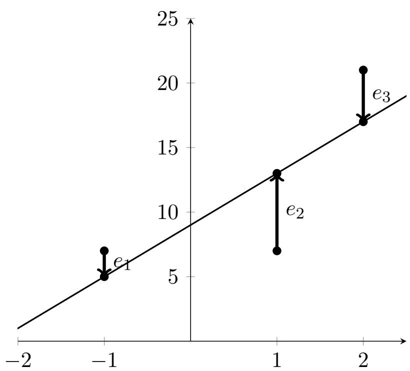


12. (a) $\vec{a}^T\vec{a}\hat{x} = \vec{a}^T\vec{b}$ becomes $n\hat{x} = \sum_{i=1}^{m} b_i$, so $\hat{x} = \frac{1}{n}\sum_{i=1}^{m} b_i$ (the average of the b_i's).

(b) $\vec{e} = \vec{b} - \vec{a}\hat{x} = \begin{pmatrix} b_1 - \hat{x} \\ \vdots \\ b_m - \hat{x} \end{pmatrix}$. So the variance $|\vec{e}|^2 = \sum_{i=1}^{m}(b_i - \hat{x})^2$, and the standard deviation is $\sqrt{\sum_{i=1}^{m}(b_i - \hat{x})^2}$.

(c) $\vec{e} = \vec{b} - \vec{p} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$. $\vec{e} \cdot \vec{p} = -2 \cdot 3 + -1 \cdot 3 + 3 \cdot 3 = 0$, so $\vec{e} \perp \vec{p}$. $P = \frac{\vec{a}\vec{a}^T}{\vec{a}\vec{a}^T} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

17. $7 = C - D$, $7 = C + D$, $21 = C + 2D$. Then $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$. So $\hat{x} = (A^TA)^{-1}A^T\vec{b} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$. The best fit line is $y = 4x + 9$.



18. $\vec{p} = A\hat{x} = \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}$. $\vec{e} = \vec{p} - \vec{b} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}$. $P\vec{e} = \vec{0}$, since by definition $\vec{e} \in N(P)$.

19. We are solving $A^TA\hat{x} = A^T\vec{c}$ for $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$, $\vec{c} = [2, -6, 4]^T$. RREF of $[A^TA|A^T\vec{c}]$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, so $\hat{x} = [0, 0]^T$. The line is therefore $y = 0$. This makes sense, since $\vec{c} \perp [-1, 1, 2]^T$, the vector of times, so the projection onto $C(A)$ is the zero vector.

25. Consider the matrix $\left(\begin{array}{cc|c} 1 & t_1 & b_1 \\ 1 & t_2 & b_2 \\ 1 & t_3 & b_3 \end{array}\right)$. The row echelon form of this is

$$\left(\begin{array}{cc|c} 1 & t_1 & b_1 \\ 0 & 1 & \frac{b_2 - b_1}{t_2 - t_1} \\ 0 & 0 & (b_3 - b_1) - \frac{b_2 - b_1}{t_2 - t_1}(t_3 - t_1) \end{array}\right)$$

So we need $(b_3 - b_1) - \frac{t_3-t_1}{t_2-t_1}(b_2 - b_1) = 0$, or

$$\frac{b_3 - b_1}{t_3 - t_1} = \frac{b_2 - b_1}{t_2 - t_1}.$$

That is, the slope between (t_1, b_1) and (t_2, b_2) must be the same as the slope between (t_1, b_1) and (t_3, b_3)....which makes sense!

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4. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Any non-square matrix with orthonormal columns will do.

   (b) Any vector and the the zero vector.

   (c) $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. There are many other answers.

6. $(Q_1Q_2)^T(Q_1Q_2) = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$.

10. (a) If $c_1\vec{q_1} + c_2\vec{q_2} + c_3\vec{q_3} = \vec{0}$, and the q_i's are orthonormal, then by taking dot product with q_1, we get $c_1\vec{q_1} \cdot \vec{q_1} + c_2\vec{q_2} \cdot \vec{q_1} + c_3\vec{q_3} \cdot \vec{q_1} = \vec{0} \cdot \vec{q_1} \Rightarrow c_1 = 0$. Similarly, by taking dot products with q_2 and q_3, we get that $c_2 = c_3 = 0$.

    (b) If $Q\vec{x} = \vec{0}$, then $Q^TQ\vec{x} = Q^T\vec{0} = \vec{0}$. Since Q is orthonormal, $Q^TQ = I$, so we get $\vec{x} = \vec{0}$.

11. (a) First normalize $\vec{a}$ to get $q_1 = \frac{1}{10}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}$. Then $v_2 = \vec{b} - \vec{q_1}^T\vec{b}\vec{q_1} = \begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}$. Normalize to get

    $q_2 = \frac{1}{10}\begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}$.

    (b) We want to project $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto the plane. Let $Q = [q_1|q_2]$. Then $P = QQ^T =$

    $\frac{1}{50}\begin{pmatrix} 25 & -9 & -12 & 20 & 0 \\ -9 & 9 & 12 & 0 & 12 \\ -12 & 12 & 16 & 0 & 16 \\ 20 & 0 & 0 & 25 & 15 \\ 0 & 12 & 16 & 15 & 25 \end{pmatrix}$. So $P\vec{v} = \frac{1}{50}\begin{pmatrix} 25 \\ -9 \\ -12 \\ 20 \\ 0 \end{pmatrix}$.

15. (a) By GS on the matrix $[A|e_1]$ where e_1 is $[1,0,0]^T$, we get $q_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-2\end{pmatrix}$, $q_2 = \frac{1}{\sqrt{66}}\begin{pmatrix}7\\1\\4\end{pmatrix}$,

$q_3 = \frac{1}{\sqrt{11}}\begin{pmatrix}1\\3\\-1\end{pmatrix}$.

(b) $q_3 \in N(A^T)$.

(c) If $Q = [q_1|q_2]$, the least squares solution is $Q^T\begin{pmatrix}1\\2\\7\end{pmatrix} = \begin{pmatrix}-11\sqrt{6}\\ \frac{37}{\sqrt{66}}\end{pmatrix}$.

20. (a) True, since if Q is orthogonal, $Q^TQ = I$, so $Q^{-1} = Q^T$, which satisfies the same.

(b) True, since for any $\vec{x} \in \mathbb{R}^2$, $|Q\vec{x}|^2 = (Q\vec{x})^T(Q\vec{x}) = \vec{x}^TQ^TQ\vec{x} = \vec{x}^T\vec{x} = |x|^2$.

23. $q_1 = a_1$, $q_2 = \frac{1}{3}a_2 - \frac{2}{3}a_1$, $q_3 = \frac{1}{5}v_3 - \frac{2}{5}v_2$. $Q = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $R = \begin{pmatrix}1&2&4\\0&3&6\\0&0&5\end{pmatrix}$.

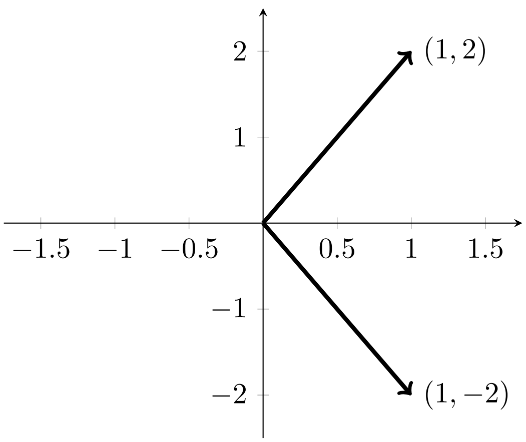
24. (a) A simple basis is $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$.

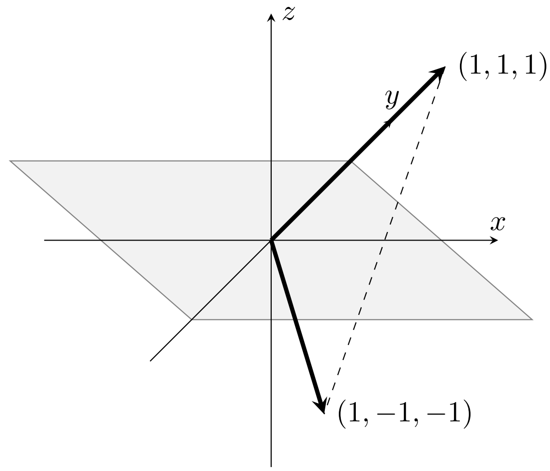
(b) Coefficients of the hyperplane form a basis: $\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}$. (Or: append $[1,0,0,0]$ to the matrix formed by the basis above and do GS to get the same result.)

(c) $\vec{b_1} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\3\end{pmatrix}$, and $\vec{b_2} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}$.

32. $Q_1 = I - 2\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} - 2\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$. $Q_2 = I - 2\begin{pmatrix}0\\\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}0&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{pmatrix} =$
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\begin{pmatrix}0&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$.

Note that Q_1 reflects in the x axis, and Q_2 reflects in the plane $y = -z$.





33. The only such matrix is the identity.