Final Exam Practice Problems

Computational exercises

1. Consider the matrix

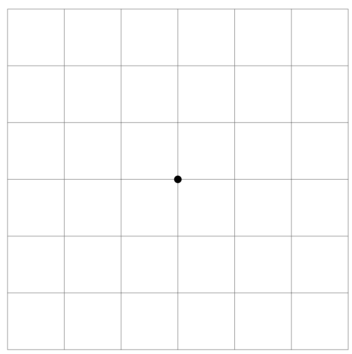
$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Use Gauss–Jordan elimination to put A into reduced row echelon form. Circle the free columns.

b) Find bases for the four fundamental subspaces of A.

c) The rank of A is $\boxed{}$.

d) Draw a picture of the column space $C(A)$ below.



e) Write down a vector $\mathbf{b}$ in $\mathbf{R}^2$ such that $A\mathbf{x} = \mathbf{b}$ has no solution. If no such vector exists, explain why not.

f) The null space is a (circle one) $\begin{pmatrix} \text{point} \\ \text{line} \\ \text{plane} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{}}$.

g) Find the general solution of $A\mathbf{x} = \binom{-2}{4}$ in parametric vector form.

2. Consider

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

a) Find the general solution of the system of equations $A\mathbf{x} = \mathbf{b}$. Express your answer in parametric vector form.

b) Compute bases for the four fundamental subspaces of A.

c) What is $\dim(N(A))$?

3. Consider

$$A = \begin{pmatrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

a) Find the reduced row echelon form of the augmented matrix $\left(A \mid b\right)$.

b) Find the parametric vector form for the solution set of $A\mathbf{x} = \mathbf{b}$.

c) What best describes the geometric relationship between the solutions of $A\mathbf{x} = 0$ and the solutions of $A\mathbf{x} = \mathbf{b}$? (Same A and $\mathbf{b}$ as above.)

(1) They are both lines through the origin.

(2) They are parallel lines.

(3) They are both planes through the origin.

(4) They are parallel planes.

4. Consider the matrix

$$D = \begin{pmatrix} 1 & 2 & 3 & 2 & 14 & 9 \\ 0 & 0 & 0 & 2 & 10 & 6 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Note that this matrix is in row echelon form.

a) Fill in the blanks:

(1) The column space $C(D)$ is a subspace of $\mathbf{R}^m$, where $m = \boxed{}$.

(2) The null space $N(D)$ is a subspace of $\mathbf{R}^n$, where $n = \boxed{}$.

b) Write down a vector $\mathbf{b}$ such that $D\mathbf{x} = \mathbf{b}$ has no solution. If no such vector exists, explain why not.

c) Compute the reduced row echelon form of D.

d) Find bases of the four fundamental subspaces of D.

5. Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) Find the parametric vector form for the general solution.

b) Find the parametric vector form of the corresponding *homogeneous* equations.

c) Find a linear dependence relation among the vectors

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right\}.$$

6. Consider the following matrix A:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \end{pmatrix}.$$

a) Find a basis for $N(A)$.

b) For each of the following vectors $\mathbf{v}$, decide if $\mathbf{v}$ is in $N(A)$, and if so, express $\mathbf{v}$ as a linear combination of the basis vectors you found above.

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

7. Consider the system below, where h and k are real numbers.

$$\begin{aligned} x + 3y &= 2 \\ 3x - hy &= k. \end{aligned}$$

a) Find the values of h and k which make the system inconsistent.

b) Find the values of h and k which give the system a unique solution.

c) Find the values of h and k which give the system infinitely many solutions.

8. Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

9. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

a) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? If not, find a linear dependence relation.

b) Give a geometric description of $S\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

c) Write $(2, 6, -2)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

10. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the subspace $W = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

a) Find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

b) What is the dimension of W?

c) Which subsets of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a basis for W?

[*Hint:* it is helpful, but not necessary, to use the fact that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal.]

11. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ h \\ 5 \end{pmatrix}.$$

a) Find the value of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly *de*pendent.

b) For this value of h, produce a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

12. Find all values of h such that $\begin{pmatrix} 1 \\ h \\ 5 \end{pmatrix}$ is *not* in the span of $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

13. Write a vector b in $\mathbf{R}^3$ which is *not* a linear combination of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

14. Consider the matrix

$$B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 7 & 1 \\ 2 & 4 & 1 \end{pmatrix}.$$

Is B invertible? If so, find its inverse. If not, explain why.

15. Consider the matrix

$$A = \begin{pmatrix} 5 & 4 & 1 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

a) Find the inverse of A.

b) Express A^{-1} as a product of elementary matrices.

c) Solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is an unknown vector.

(Your answer will be a formula in b_1, b_2, b_3.)

16. Compute the matrix A satisfying

$$A\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

17. Let

$$A=\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}5&6&7&8\\6&8&10&12\\-9&-10&-11&-12\end{pmatrix}.$$

a) What three row operations are needed to transform A into B?

b) What are the elementary matrices E_1, E_2, E_3 for these three operations?

c) Write an equation for B in terms of A and E_1, E_2, E_3.

d) Write an equation for A in terms of B and E_1, E_2, E_3.

18. Consider the matrix

$$A=\begin{pmatrix}2&3&1\\-4&-5&-3\\-2&-6&0\end{pmatrix}.$$

a) Find a lower-triangular matrix L with ones on the diagonal and an upper-triangular matrix U such that $A = LU$.

b) Solve the equation $A\mathbf{x}=\begin{pmatrix}0\\0\\4\end{pmatrix}$ using the LU decomposition you found in (a).

19. Consider the matrix

$$A=\begin{pmatrix}1&2&5&0\\1&2&4&2\\0&-1&0&8\\-1&-3&-1&-1\end{pmatrix}.$$

a) Find a permutation matrix P, a lower-triangular matrix L with ones on the diagonal, and an upper-triangular matrix U such that $PA = LU$.

b) Use part (a) to solve $A\mathbf{x}=\mathbf{b}$, for $\mathbf{b}=(12,1,-30,6)$.

20. Find the general solution of the equation $A\mathbf{x}=\mathbf{b}$ using the $PA = LU$ factorization given below. Do not compute A.

$$P=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad L=\begin{pmatrix}1&0&0\\-1&1&0\\-2&-1&1\end{pmatrix} \quad U=\begin{pmatrix}1&0&1&-1\\0&1&0&4\\0&0&1&2\end{pmatrix} \quad \mathbf{b}=\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

21. Let $\mathbf{v}, \mathbf{w}$ be orthogonal vectors with $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = \sqrt{2}$. Let $\mathbf{x} = -\mathbf{v} + 3\mathbf{w}$ and $\mathbf{y} = 3\mathbf{v} - \mathbf{w}$. Compute:

$$\|\mathbf{x}\| = \boxed{} \qquad \mathbf{x} \cdot \mathbf{y} = \boxed{}.$$

22. Let $W = S\{(2, 1, -6, 18)\}$. Compute an orthogonal basis of $W^{\perp}$.

23. Let W be the set of all vectors of the form $(x, y, x + y)$. Compute an orthonormal basis of $W^{\perp}$.

24. Find an orthonormal basis of $\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \\ -2 \\ 0 \end{pmatrix}\right\}$.

25. Consider the subspace $V = \left\{\text{all solutions of } \begin{array}{r} x_1 + x_2 + 7x_3 + 5x_4 = 0 \\ -2x_1 - 2x_2 + 4x_3 + 8x_4 = 0 \end{array}\right\}$.

a) Find an orthonormal basis of V.

b) Find an orthonormal basis of $V^{\perp}$.

c) Compute the matrix P for projection onto V.

d) Compute the matrix $P_{\perp}$ for projection onto $V^{\perp}$.

e) Compute the projection $\mathbf{p}$ of $\mathbf{v} = (1, 0, 0, 1)$ onto $V^{\perp}$.

26. Consider the subspace $V = \{\text{all solutions of } 2x + 3y + 4z = 0\}$.

a) Find a basis for V.

b) Find a basis for $V^{\perp}$.

27. Let W be the span of $(1, 1, 1, 1)$ in $\mathbf{R}^4$. Find a matrix whose null space is $W^{\perp}$.

28. If $W = S\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right\}$ then $\dim(W^{\perp}) = \boxed{}$.

29. Find two linearly independent vectors that are orthogonal to $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

30. Find a nonzero vector orthogonal to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

31. Let A be a 4×5 matrix whose null space is spanned by $\mathbf{a} = (1, 2, 1, 3, 1)$.

a) Find the projection matrix onto the null space of A.

b) Find the projection matrix onto the row space of A.

32. Let W be the span of the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

a) Find the orthogonal projection of $\mathbf{y} = (2, -1, 3)$ onto W.

b) How far is $\mathbf{y}$ from W?

33. Let L be the line $y = x$ in $\mathbf{R}^2$.

a) Compute the matrices P and $P_\perp$ for orthogonal projection onto L and $L^\perp$, respectively.

b) Find a basis for $C(P)$ without using elimination.

c) Find a basis for $C(PP_\perp)$ without using elimination.

34. Consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}.$$

Compute the distance from $\mathbf{z}$ to $C(A)$.

35. Find the least-squares solution of

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 0 \end{pmatrix}.$$

36. Consider the QR decomposition

$$A = QR = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{2} \\ -1 & 0 \\ -1 & 0 \\ 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 0 & 2\sqrt{2} \end{pmatrix}.$$

Determine the least-squares solution of $A\mathbf{x} = (0, 2, 0, 0)$.

37. Consider the matrix

$$A = \begin{pmatrix} 3 & -3 \\ 3 & 1 \\ 3 & 5 \\ 3 & 1 \end{pmatrix}.$$

a) Use Gram–Schmidt to find the QR decomposition of A.

b) Find the least squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x} = (2,4,6,8)$, using your QR decomposition above or otherwise.

c) Find the projection $\mathbf{p}$ of $\mathbf{v}$ onto $C(A)$.

38. Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

a) Find the QR decomposition of A.

b) Find the least squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x} = (1,2,3,5)$, using your QR decomposition above.

39. Consider the matrix

$$A = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{pmatrix}.$$

a) Find an orthonormal basis for $C(A)$.

b) Find a QR factorization of A.

c) Find a different orthonormal basis for $C(A)$. (Reordering and scaling your basis in (a) does not count.)

40. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}.$$

a) Find the QR factorization of A.

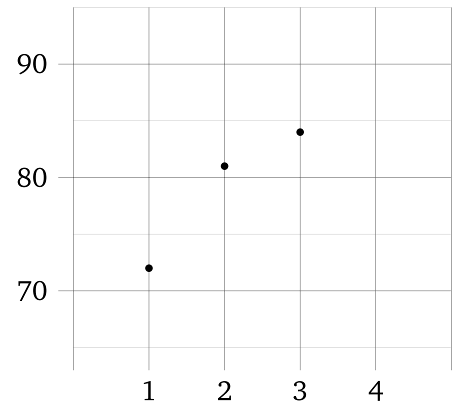
b) Find the least-squares solution of the system $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ using the QR factorization you found in (a).

c) Compute the matrix P for the projection onto $C(A)$ using the QR factorization you found in (a).

41. Your roomate Karxon is currently taking Math 105L. Karxon scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Karxon does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) The general equation of a line in $\mathbf{R}^2$ is $y = Cx + D$. Write down the system of linear equations in C and D that would be satisfied by a line passing through the points $(1,72)$, $(2,81)$, and $(3,84)$, and then write down the corresponding matrix equation.

b) Solve the corresponding least squares problem for C and D, and use this to *write down* and *draw* the the best fit line below. [Use a calculator]



$$y = \square\, x + \square$$

c) What score does this line predict for the fourth (final) exam?

42. In this problem, you will find the best-fit line through the points $(0,6)$, $(1,0)$, and $(2,0)$.

a) The general equation of a line in $\mathbf{R}^2$ is $y = C + Dx$. Write down the system of linear equations in C and D that would be satisfied by a line passing through all three points, then write down the corresponding matrix equation.

b) Solve the least squares problem in (a) for C and D. Give the equation for the best fit line, and graph it along with the three points.

43. Compute determinants of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \begin{pmatrix} 5 & 0 & 0 \\ -3 & 0 & 0 \\ 8 & 5 & -1 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 7 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & -6 \\ 1 & 0 & -3 \end{pmatrix}$$

44. Let A be the 3×3 matrix satisfying

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Compute $\det(A)$.

45. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^2.$$

46. Given that

$$\det\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 5,$$

compute the determinants of the following matrices:

$$\begin{pmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ 5 & 7 & 9 \\ 1 & 2 & 3 \end{pmatrix}.$$

47. Find the volume of the parallelepiped defined by the columns of this matrix:

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

48. For which value(s) of k, if any, do these vectors not form a basis of $\mathbf{R}^4$?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -6 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ k \end{pmatrix}$$

49. Find all values of k so that the following set of vectors is linearly dependent.

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ k \\ -7 \end{pmatrix} \right\}$$

50. Let A, B, C be 4×4 matrices with $\det(A) = 2$, $\det(B) = -3$, $\det(C) = 5$. Find the determinants of these matrices:

$$AB \qquad AC^{-1}B \qquad B^T C^2 \qquad A^3 B^{-1} C^T \qquad 4C.$$

51. For which value(s) of a is $\lambda = 1$ an eigenvector of this matrix?

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

52. Let A be a matrix with characteristic polynomial

$$p(\lambda) = (3-\lambda)(1-\lambda)^2(4+\lambda)^3.$$

a) A is a $\square \times \square$ matrix.

b) $\det(A) = \square$ and $\mathrm{Tr}(A) = \square$.

c) What are the eigenvalues of A?

d) The rank of A is $\square$ and $\dim N(A) = \square$.

e) Do we have enough information to decide if A is diagonalizable?

53. Consider these 3×3 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}.$$

For each matrix,

a) Compute all eigenvalues with algebraic multiplicity.

b) Compute the geometric multiplicity of each eigenvalue.

c) If the matrix is diagonalizable, express it in the form $X\Lambda X^{-1}$ for Λ diagonal. Otherwise, explain why the matrix is not diagonalizable.

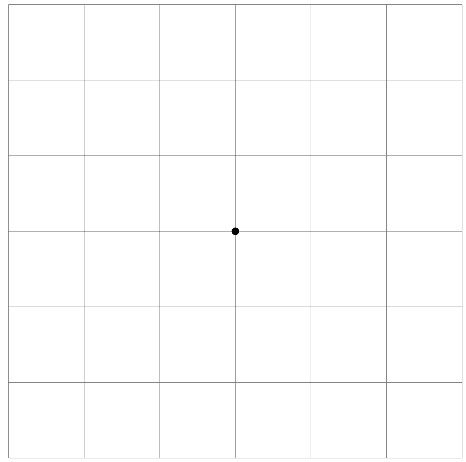
54. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

a) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

b) Compute $A^n \mathbf{v}_0$ for $\mathbf{v}_0 = \binom{1}{2}$. What happens when $n \to \infty$?

c) In the diagram, *draw and label* the eigenspaces of A, and draw the vectors $\mathbf{v}_0, A\mathbf{v}_0, A^2\mathbf{v}_0, A^3\mathbf{v}_0, \ldots$ as points. (The grid lines are one unit apart, and the dot is the origin.) [Hint: you do not have to compute $A^n\mathbf{v}_0$ numerically to do this.]



d) Solve the system of ordinary differential equations

$$\begin{aligned} \tfrac{d}{dt}u_1 &= 2u_1 - u_2 & u_1(0) &= 1 \\ \tfrac{d}{dt}u_2 &= \tfrac{3}{2}u_1 - \tfrac{1}{2}u_2 & u_2(0) &= 2 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} u_1(t) &= \\ u_2(t) &= \end{aligned}$$

55. Consider the sequence of numbers $0, 1, 5, 31, 185, \ldots$ given by the recursive formula

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= 5a_{n-1} + 6a_{n-2} \qquad (n \geq 2). \end{aligned}$$

a) Find a matrix A such that

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

for all $n \geq 2$.

b) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

c) Give a formula for A^n. Your answer should be a single matrix whose entries depend only on n.

d) Give a non-recursive formula for a_n.

56. Solve the system of ordinary differential equations

$$\begin{aligned} u_1' &= 3u_1 + 24u_2 & u_1(0) &= 7 \\ u_2' &= -2u_1 - 11u_2 & u_2(0) &= -2. \end{aligned}$$

What happens to $u_1(t)$ and $u_2(t)$ as $t \to \infty$?

57. Solve the system of ordinary differential equations

$$\begin{aligned} u_1' &= u_2 & u_1(0) &= 1 \\ u_2' &= -u_1 & u_2(0) &= 1. \end{aligned}$$

Your answer should not contain any complex numbers.

58. Consider the matrix
$$A = \begin{pmatrix} -2\sqrt{3}-1 & 5 \\ -1 & -2\sqrt{3}+1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

c) Express A as a product $X\Lambda X^{-1}$, where X is invertible and Λ is diagonal. These matrices will have complex entries.

d) Solve the system of ordinary differential quations
$$\mathbf{u}' = A\mathbf{u} \qquad \mathbf{u}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$
Your answer should not contain any complex numbers.

e) What happens to $u_1(t)$ and $u_2(t)$ as $t \to \infty$?

59. Consider the symmetric matrix
$$A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$
Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

60. Consider the matrix
$$A = \begin{pmatrix} 23 & 36 \\ 36 & 2 \end{pmatrix}.$$

a) Find the eigenvalues of this matrix.

b) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

c) Consider the following system of differential equations with initial values:
$$\begin{aligned} \tfrac{dx}{dt} &= 23x + 36y \qquad & x(0) &= 7 \\ \tfrac{dy}{dt} &= 36x + 2y \qquad & y(0) &= -1. \end{aligned}$$
Find the solutions $x(t)$ and $y(t)$.

d) Find a formula for A^n in terms of n.

61. Consider the matrix
$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

a) Compute the eigenvalues of A. What are their algebraic and geometric multiplicities?

b) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

c) Compute A^{114} and A^{115}.

62. Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

a) Find an orthogonal matrix Q and a diagonal matrix Λ such that $S = Q\Lambda Q^T$.

b) Circle one: S is

positive-definite   positive-semidefinite   neither of these

c) Write down the singular value decomposition of S.

d) Find a different orthogonal matrix $Q' \neq Q$ and a different diagonal matrix $\Lambda' \neq \Lambda$ such that $S = Q'\Lambda' Q'^T$.

63. Consider the quadratic form

$$q(x,y,z) = 9x^2 + 10y^2 + 8z^2 + 4xy - 4xz.$$

a) Construct a symmetric matrix S such that $q(\mathbf{x}) = \mathbf{x}^T S\mathbf{x}$.

b) Find $\mathbf{x}$ maximizing $\|\mathbf{x}\|$ subject to the constraint $q(\mathbf{x}) = 1$.
[Hint: one of the eigenvalues of S is 12.]

64. Compute the singular value decompositions of these matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ -7 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 1 & -2 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

65. Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

From this you can read off all of the following properties of A.

a) A is a ☐ × ☐ matrix of rank $r =$ ☐.

b) Find orthonormal bases of the four fundamental subspaces of A.

c) Express A as a linear combination of rank-one matrices $\mathbf{u}\mathbf{v}^T$ (your answer should consist of vectors with numbers, not letters).

d) Find a *unit* vector $\mathbf{x}$ maximizing $\|A\mathbf{x}\|$.

e) Compute the matrix P for orthogonal projection onto $C(A)$ (write it as a product, without expanding it out).

66. A certain matrix A has singular value decomposition $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \\ | & | & | & | \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{pmatrix}.$$

a) What is the rank of A?

b) What is the maximum value of $\|A\mathbf{x}\|$ subject to $\|\mathbf{x}\| = 1$?

c) Find orthonormal bases of the four fundamental subspaces of A.

d) What is the singular value decomposition of A^T?

67. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

This question is about the singular value decomposition $A = U\Sigma V^T$. Let $U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$ and $V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$.

a) Find $\mathbf{v}_1$ *without* computing A^TA or AA^T. [Hint: in which space does $\mathbf{v}_1$ live?]

b) Using your answer to (a), compute σ_1 and $\mathbf{u}_1$.

c) Write down the singular value decomposition $A = U\Sigma V^T$ of A.

d) Compute the pseudo-inverse A^+.

e) Find the shortest least-squares solution of $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Conceptual exercises

68. Which of these matrices are in reduced row echelon form?

$$\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1\\0&0\end{pmatrix} \quad \left(\begin{array}{cccc|c}1&1&0&-3&1\\0&0&1&-1&5\\0&0&0&0&0\end{array}\right)$$

69. Decide if the following sets of vectors span a point, line, plane, or all of $\mathbf{R}^3$. [Hint: you should be able to eyeball these.]

$$\left\{\begin{pmatrix}1\\-2\\5\end{pmatrix}, \begin{pmatrix}4\\-8\\20\end{pmatrix}\right\} \qquad \left\{\begin{pmatrix}1\\-2\\5\end{pmatrix}, \begin{pmatrix}4\\2\\0\end{pmatrix}, \begin{pmatrix}1\\1/2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$$

70. Let A be a 2×2 matrix such that $N(A)$ is the line $y = x$. Let $\mathbf{b}$ be a *nonzero* vector in $\mathbf{R}^2$. Which of the following are definitely *not* the solution set of $A\mathbf{x} = \mathbf{b}$? (Circle all that apply.)

(1) The line $y = x$.
(2) The y-axis.
(3) The line $y = x + 1$
(4) The set $\{0\}$.
(5) The empty set.

71. Which of the following properties of a matrix A are preserved under row operations? (In other words, which remain unchanged after doing any row operation?)

(1) The rank
(2) The solutions of $A\mathbf{x} = \mathbf{b}$
(3) The dimension of the null space
(4) The column space
(5) The null space
(6) The row space
(7) The left null space
(8) The eigenvalues
(9) The eigenvectors
(10) The singular values
(11) The determinant
(12) The reduced row echelon form

72. Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "the columns of A are linearly independent?"

(1) A has full column rank
(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(3) $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(4) $A\mathbf{x} = \mathbf{0}$ has a unique solution
(5) A has n pivots
(6) $N(A) = \{\mathbf{0}\}$

(7) $m \geq n$
(8) $A^T A$ is invertible
(9) AA^T is invertible
(10) $A^+ A$ is the identity matrix
(11) The row space of A is $\mathbf{R}^n$

73. Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "A has full row rank?"

(1) The rows of A are linearly independent
(2) The left null space of A is $\{\mathbf{0}\}$
(3) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(4) $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(5) A has m pivots
(6) $C(A) = \mathbf{R}^m$
(7) $n \geq m$
(8) $A^T A$ is invertible
(9) AA^T is invertible
(10) AA^+ is the identity matrix
(11) The row space of A is $\mathbf{R}^n$

74. Consider the subspace

$$V = S\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}.$$

Find two other vectors that span V. (You may not include scalar multiples of the vectors $\begin{pmatrix}1\\1\end{pmatrix}$ or $\begin{pmatrix}1\\-1\end{pmatrix}$.)

75. Write three different nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbf{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly *de*pendent, but $\mathbf{v}_3$ is *not* in $S\{\mathbf{v}_1, \mathbf{v}_2\}$. Clearly indicate which is $\mathbf{v}_3$.

76. If A is a 20×60 matrix, then $\text{rank}(A) + \dim N(A) = \boxed{}$.

77. If A is a 5×6 matrix of rank 2, then $N(A)$ is a $\boxed{}$-dimensional subspace of $\mathbf{R}^{\boxed{}}$.

78. Which of the following are subspaces of $\mathbf{R}^4$ and **why?**

a) Span $\left\{\begin{pmatrix}1\\0\\3\\2\end{pmatrix}, \begin{pmatrix}-2\\7\\9\\13\end{pmatrix}, \begin{pmatrix}144\\0\\0\\1\end{pmatrix}\right\}$

b) $N\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

c) $C\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

d) $V = \left\{ \text{all vectors } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \text{ such that } xy = zw \right\}$

79. Consider the subspace V of $\mathbf{R}^4$ consisting of all vectors (x, y, z, w) such that $w = 0$.

a) Explain why $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for V.

b) Explain why $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for V.

80. Which of the following are subspaces of $\mathbf{R}^n$?

(1) The null space of an $m \times n$ matrix
(2) An eigenspace of an $n \times n$ matrix (for a particular eigenvalue)
(3) The column space of an $m \times n$ matrix
(4) The span of $n - 1$ vectors in $\mathbf{R}^n$
(5) The row space of an $m \times n$ matrix
(6) $W^\perp$ where W is a subspace of $\mathbf{R}^n$

81. Let A be an invertible $n \times n$ matrix. Which of the following can you conclude?

(1) The columns of A span $\mathbf{R}^n$
(2) $\det(A) \neq 0$
(3) A is diagonalizable
(4) The rank of A equals n
(5) $N(A) = \{\mathbf{0}\}$
(6) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^n$
(7) The eigenvalues of A are positive
(8) There exists a matrix B such that $AB = I_n$

82. Let A be an $n \times n$ matrix that is *not* invertible. Which of the following can you conclude?

(1) A has two identical columns
(2) $\det(A) = 0$
(3) A has a row of zeros
(4) The rank of A equals zero
(5) There are two different vectors $\mathbf{x}, \mathbf{y}$ in $\mathbf{R}^n$ such that $A\mathbf{x} = A\mathbf{y}$

(6) Zero is an eigenvalue of A
(7) A is not diagonalizable
(8) The rank of A is less than n
(9) A has linearly dependent rows

83. Give an example of a 3×2 matrix A and a vector $\mathbf{b}$ in $\mathbf{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has more than one least-squares solution.

84. Let A be an $n \times n$ matrix. Which of the following are equivalent to the statement "A is diagonalizable over the real numbers?"
(1) A is similar to a diagonal matrix.
(2) A has at least one eigenvector for each eigenvalue.
(3) For each real eigenvalue λ of A, the dimension of the λ-eigenspace is equal to the algebraic multiplicity of λ.
(4) A has n linearly independent eigenvectors.
(5) A is invertible.

85. Let A be a square matrix. Complete the following definitions, paying attention to the quantifiers (there exists / for all):

a) An eigenvector of A is...

b) An eigenvalue of A is...

86. Let A be an $n \times n$ matrix with eigenvalues 2 and 3. Compute the eigenvalues of these matrices:

$$A^{-1} \qquad A - 7I_n \qquad 2A.$$

87. Let V be a subspace of $\mathbf{R}^n$; assume that $V \neq \{0\}$ and $V \neq \mathbf{R}^n$. Let P be the matrix for the projection onto V.

a) Show that the eigenvalues of P are 0 and 1.

b) Which of the four fundamental subspaces is equal to the 0-eigenspace? Which of the four fundamental subspaces is equal to the 1-eigenspace?

c) Why is P diagonalizable? What diagonal matrix is it similar to?

88. Without computing any eigenvalues, decide which of the following matrices are positive definite.

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 2 \\ 3 & -1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

89. Let A be a 3×3 matrix with singular values 1 and 2. Compute the following quantities:

a) The rank of A.

b) The determinant of A.

c) The determinant of $A^T A$.

d) The eigenvalues of $A^T A$.

90. Indicate which of the following phrases and expressions are not defined.

(1) The dimension of the matrix A is equal to 5.
(2) The subspace V has dimensions 4×4.
(3) The matrix A is linearly independent.
(4) The subspace V has basis $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.
(5) The quantity $\mathbf{u} \cdot \mathbf{u} \cdot \mathbf{u}$, for $\mathbf{u}$ in $\mathbf{R}^n$.
(6) $A^\perp$ for A an $m \times n$ matrix.

91. True/false problems:

(1) If $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$, then A has full row rank.
(2) If the columns of A span $\mathbf{R}^m$, then $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbf{R}^m$.
(3) If $A\mathbf{x} = \mathbf{b}$ is consistent, then $A\mathbf{x} = 5\mathbf{b}$ is consistent.
(4) If $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$, then every vector in $S\{\mathbf{x}\}$ is also a solution of $A\mathbf{x} = \mathbf{b}$.
(5) The solution set of $A\mathbf{x} = \mathbf{b}$ is a subspace.
(6) The solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace.
(7) The columns of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are linearly independent.
(8) The matrix equation $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 3 \end{pmatrix}\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbf{R}^2$.
(9) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbf{R}^5$, then $n = 5$.
(10) Any four linearly independent vectors in $\mathbf{R}^4$ form a basis of $\mathbf{R}^4$.
(11) There exists a 3×5 matrix of rank 4.
(12) There exists a 3×5 matrix whose null space has dimension 4.
(13) Every elementary matrix is invertible.
(14) If V is the set of all vectors (x, y, z) in $\mathbf{R}^3$ such that $x = 2y$, then the orthogonal complement of V is a line.
(15) If A is a 3×4 matrix, then $C(A)^\perp$ is a subspace of $\mathbf{R}^4$.
(16) If A has the QR factorization $A = QR$, then the columns of R span $C(A)$.
(17) If A has the QR factorization $A = QR$, then the columns of Q span $C(A)$.
(18) A least-squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ minimizes the quantity $\|A\widehat{x} - \mathbf{b}\|^2$.
(19) If $\mathbf{x}$ is in a subspace V, then the projection of $\mathbf{x}$ onto V is equal to zero.
(20) If $\mathbf{x}$ is not in a subspace V, then the projection of $\mathbf{x}$ onto $V^\perp$ is nonzero.
(21) If V is a two-dimensional subspace of $\mathbf{R}^4$, then $\dim(V^\perp) = 2$.
(22) Every subspace of $\mathbf{R}^n$ has an orthonormal basis.
(23) If A is an $n \times n$ matrix and c is a scalar, then $\det(cA) = c\det(A)$.

(24) If A is a matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + \lambda^2 + \lambda$, then A is invertible.
(25) If A is a square matrix, then the nonzero vectors in $N(A)$ are eigenvalues of A.
(26) The eigenvalues of a triangular square matrix are the numbers on the main diagonal.
(27) If λ is a complex eigenvalue of a real matrix A, then so is $\overline{\lambda}$.
(28) A triangular matrix A with real entries can have a complex (non-real) eigenvalue.
(29) If A is an $n \times n$ matrix and $\det(A) = 2$, then 2 is an eigenvalue of A.
(30) The $n \times n$ zero matrix is diagonalizable.
(31) If A is a 3×3 matrix and $1, 2, 3$ are eigenvalues of A, then A is diagonalizable.
(32) A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.
(33) If S is a symmetric matrix with eigenvalue λ, then the algebraic multiplicity of λ equals the geometric multiplicity.
(34) The maximum value of $\|A\mathbf{x}\|/\|\mathbf{x}\|$, for $\mathbf{x} \neq 0$, is the largest eigenvalue of A.

92. In the following, if the statement is true, prove it; if not, give a counterexample.
(1) A system of linear equations $A\mathbf{x} = \mathbf{b}$ cannot have exactly two solutions.
(2) If a linear system has more equations than unknowns, then the system cannot have a unique solution.
(3) If a linear system has more unknowns than equations, then the system cannot have a unique solution.
(4) If A has more columns than rows, then the columns of A must be linearly dependent.
(5) If A has more rows than columns, then the columns of A must be linearly independent.
(6) If A has more rows than columns, then $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbf{R}^m$.
(7) The zero vector is in the span of the columns of A.
(8) $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the span of the columns of A.
(9) If $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$ in $\mathbf{R}^m$, then $A\mathbf{x} = \mathbf{0}$ has a unique solution.
(10) If A is an $m \times n$ matrix with linearly dependent columns, then the columns of A do not span $\mathbf{R}^m$.
(11) If A and B are matrices and AB is defined, then the column space of AB is contained in the column space of A.
(12) Any two vectors in $\mathbf{R}^3$ span a plane.
(13) If $A\mathbf{x} = \mathbf{0}$ has only one solution, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^m$.
(14) There is no 3×3 matrix A such that $C(A) = N(A)$.
(15) Any set of vectors containing $\mathbf{0}$ is linearly dependent.
(16) If A is a 3×3 matrix of rank 2, then $A^2 \neq 0$.
(17) For any matrix A, we have $N(A) = N(A^T A)$.
(18) If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\{\mathbf{v}_2 \mathbf{v}_3\}$ is linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(19) A singular matrix A must have a zero entry.
(20) An invertible matrix is a product of elementary matrices.
(21) An invertible matrix has a factorization $A = LU$, for U upper-triangular and L lower-triangular.
(22) If A, B, C are nonzero matrices such that $AB = AC$, then $B = C$.
(23) If A, B, C are invertible matrices such that $AB = AC$, then $B = C$.
(24) If A and B are nonzero matrices and AB is defined, then the column space of AB is equal to the column space of A.
(25) If A and B are $n \times n$ matrices and A is singular, then the columns of AB are linearly dependent.
(26) If A is an invertible $n \times n$ matrix, then the columns of A^{-1} span $\mathbf{R}^n$.
(27) If A is an $m \times n$ matrix, then $\dim C(A) + \dim N(A) = n$.
(28) If V and W are subspaces of $\mathbf{R}^n$ with $\dim(V) + \dim(W) > n$, then there is a nonzero vector contained in both V and W.
(29) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^{\perp}$.
(30) Every projection matrix P can be written as QQ^T for a matrix Q with orthonormal columns.
(31) If P is the projection matrix onto the the row space of a matrix A, then $I - P$ projects vectors onto the null space of A.
(32) The row space of A equals the row space of $A^T A$.
(33) Let W be a subspace of $\mathbf{R}^n$. If $\mathbf{y}$ is in W and $\mathbf{y}$ is in $W^{\perp}$, then $\mathbf{y} = \mathbf{0}$.
(34) Let W be a subspace of $\mathbf{R}^n$ and let $\mathbf{x}$ be a vector in $\mathbf{R}^n$. Then $\mathbf{x}$ can be expressed in the form $\mathbf{y} + \mathbf{z}$ for $\mathbf{y}$ in W and $\mathbf{z}$ in $W^{\perp}$.
(35) If P is a projection matrix then $P^2 = P$.
(36) If A and B are orthogonal square matrices, then so is $A + B$.
(37) If A and B are orthogonal square matrices, then so is AB.
(38) A matrix with orthonormal columns has full column rank.
(39) If Q is an orthogonal $n \times n$ matrix and $\mathbf{x}, \mathbf{y}$ are vectors in $\mathbf{R}^n$, then $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ and $||Q\mathbf{x}|| = ||\mathbf{x}||$.
(40) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in $\mathbf{R}^3$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal.
(41) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are nonzero orthogonal vectors in $\mathbf{R}^3$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
(42) If Q has orthonormal columns, then QQ^T is the matrix for the orthogonal projection onto $C(Q)$.
(43) If $\mathbf{b}$ is in $C(A)$, then the least-squares solutions of $A\mathbf{x} = \mathbf{b}$ are simply the solutions of $A\mathbf{x} = \mathbf{b}$.
(44) If A has linearly independent columns, then $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.
(45) If A is a 3×2 matrix with orthonormal columns $\mathbf{v}_1, \mathbf{v}_2$, then the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is $(\mathbf{b} \cdot \mathbf{v}_1, \mathbf{b} \cdot \mathbf{v}_2)$.
(46) If a matrix has determinant zero, then two of the columns (or rows) are multiples of each other, or one of the columns (or rows) is zero.
(47) If A^n is invertible for some positive integer n, then A is invertible.

(48) An orthogonal matrix has determinant ± 1.
(49) If λ is an eigenvalue of A, then the set of all eigenvectors with eigenvalue λ is a subspace.
(50) A square matrix has the same eigenvalues as its transpose.
(51) A square matrix has the same characteristic polynomial as its transpose.
(52) Every square matrix admits at least one (potentially complex) eigenvalue.
(53) If A is an $n \times n$ matrix and $A-3I$ has rank n, then 3 is not an eigenvalue of A.
(54) If A is an invertible matrix and 2 is an eigenvalue of A, then $1/2$ is an eigenvalue of A^{-1}.
(55) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A with eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.
(56) If A is a square matrix and $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A, then $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of A.
(57) Suppose A and B are $n \times n$ matrices. If 0 is an eigenvalue of AB, then 0 is an eigenvalue of BA.
(58) If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2.
(59) If λ is an eigenvalue of two $n \times n$ matrices A and B, then λ^2 is an eigenvalue of AB.
(60) If A is square and $\mathbf{v}_1, \mathbf{v}_2$ are nonzero vectors satisfying $A\mathbf{v}_1 = 2\mathbf{v}_1$ and $A\mathbf{v}_2 = 3\mathbf{v}_2$, then $\mathbf{v}_1 \perp \mathbf{v}_2$.
(61) If A is a 3×3 matrix that has eigenvalues 1 and -1, both of algebraic multiplicity one, then A is diagonalizable (over the real numbers).
(62) If A is a 3×3 matrix that has eigenvalues 1 and -1, with algebraic multiplicities 1 and 2, respectively, then A is diagonalizable.
(63) If A is a 9×9 matrix with three distinct eigenvalues, and the eigenspace corresponding to one of these eigenvalues has dimension 7, then A is diagonalizable.
(64) If $A = XBX^{-1}$ then A and B have the same characteristic polynomial.
(65) If $A = XBX^{-1}$ then A and B have the same eigenvectors.
(66) Every square matrix is diagonalizable if we allow complex eigenvalues and eigenvectors.
(67) If A is an $n \times n$ matrix with n linearly independent eigenvectors, then A^T also has n linearly independent eigenvectors.
(68) If A is diagonalizable then so is A^2.
(69) If A^2 is diagonalizable then so is A.
(70) If A is invertible and diagonalizable then so is A^{-1}.
(71) If A is a diagonalizable matrix whose only eigenvalue is 1, then A is the identity.
(72) A diagonalizable $n \times n$ matrix has n distinct eigenvalues.
(73) If S is symmetric, then either S or $-S$ is positive-semidefinite.
(74) If λ is an eigenvalue of AA^T, then λ is an eigenvalue of A^TA.
(75) If λ is a eigenvalue of A^TA and $\lambda > 0$, then λ is also an eigenvalue of AA^T.
(76) If S and T are positive definite, then so is $S + T$.
(77) If S is a symmetric $n \times n$ matrix and $\mathbf{x}, \mathbf{y}$ are vectors in $\mathbf{R}^n$, then $(S\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (S\mathbf{y})$.
(78) If S is symmetric and $\mathbf{v}_1, \mathbf{v}_2$ are nonzero vectors satisfying $S\mathbf{v}_1 = 2\mathbf{v}_1$ and $S\mathbf{v}_2 = 3\mathbf{v}_2$, then $\mathbf{v}_1 \perp \mathbf{v}_2$.
(79) If A is any matrix, then A^TA is positive semidefinite.

(80) A positive definite matrix must have positive numbers on the main diagonal.
(81) The singular values of a diagonalizable, invertible, square matrix are the absolute values of the eigenvalues. [Hint: try a 2×2 matrix of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.]
(82) If A has linearly independent columns, then A^+A is the identity matrix.
(83) The largest singular value of an orthogonal matrix is 1.
(84) The maximum value of $\|A\mathbf{x}\|$, for $\|x\| = 1$, is the largest singular value of A.
(85) The singular values of a symmetric matrix are the absolute values of the nonzero eigenvalues.

93. Give examples of matrices with the following properties. If no such matrix exists, explain why. All matrices must have real entries.

(1) A 4×4 matrix A such that $C(A) = N(A)$.
(2) A 3×2 matrix with singular values 2 and 1.
(3) A matrix having eigenvalue 3 with algebraic multiplicity 2 and geometric multiplicity 1.
(4) A matrix having eigenvalue 3 with algebraic and geometric multiplicity 2.
(5) A matrix having eigenvalue 3 with algebraic multiplicity 1 and geometric multiplicity 2.
(6) A 2×2 matrix that is invertible but not diagonalizable.
(7) A 2×2 matrix that is diagonalizable but not invertible.
(8) A 2×2 matrix that is diagonalizable and invertible.
(9) A 2×2 matrix that is neither diagonalizable nor invertible.
(10) A 3×4 matrix A and a 4×3 matrix B such that AB is invertible.
(11) A 3×4 matrix A and a 4×3 matrix B such that BA is invertible.
(12) A 3×3 symmetric matrix that is positive semidefinite but not positive definite.
(13) A 3×3 symmetric matrix that is not diagonalizable.
(14) A 3×3 matrix whose entries are all either 1 or -1, with determinant 4.
(15) A 4×6 matrix of rank 6.
(16) An invertible 2×2 matrix A such that $A\binom{1}{2} = A\binom{2}{1}$.
(17) A 2×2 matrix whose column space is the line $3x + y = 0$ and whose null space is the line $5x + y = 0$.
(18) A 2×2 matrix whose column space is the line $3x + y = 0$ and whose null space is the line $5x + y = 1$.
(19) A 2×2 matrix whose column space is the line $3x + y = 0$ and whose null space is $\{\mathbf{0}\}$.
(20) A 3×3 matrix with no real eigenvalues.
(21) A 2×2 matrix with no real eigenvalues.
(22) A 2×2 singular matrix with eigenvalue $2 + 3i$.
(23) A diagonalizable 3×3 matrix with exactly two distinct eigenvalues.
(24) A 2×2 matrix with eigenvalues 1 and 2.
(25) A 2×2 matrix with eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$ having the same eigenvalue.
(26) A nonzero 2×2 diagonalizable matrix with characteristic polynomial $p(\lambda) = \lambda^2$.
(27) A 2×2 matrix whose 1-eigenspace is the line $x + 2y = 0$ and whose 2-eigenspace is the line $x + 3y = 0$.

(28) A 4×4 matrix whose columns form an orthonormal basis for $\mathbf{R}^4$, other than the identity matrix.
(29) A symmetric, orthogonal 2×2 matrix, other than the identity matrix.
(30) A matrix A satisfying

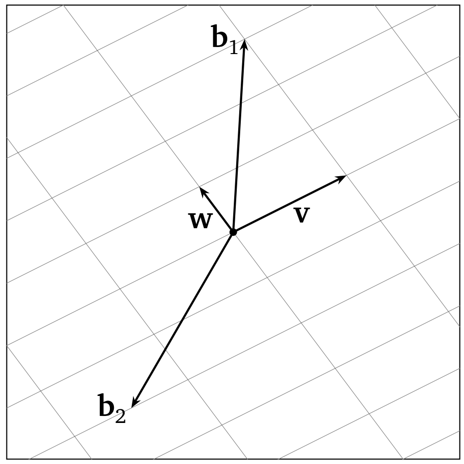
$$\dim\big(\mathrm{Row}(A)^{\perp}\big) = 2 \quad \text{and} \quad \dim\big(C(A)^{\perp}\big) = 3.$$

(31) A matrix that does not have a singular value decomposition.
(32) A 3×4 matrix with orthonormal columns.
(33) A 4×3 matrix with orthonormal columns.
(34) A symmetric matrix S satisfying

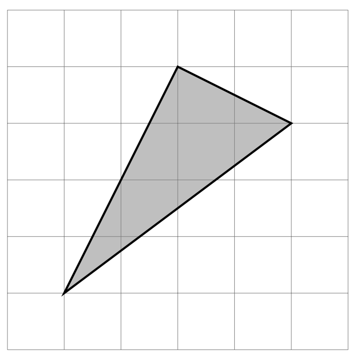
$$S\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix} \quad \text{and} \quad S\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\\0\end{pmatrix}.$$

Pictures

94. A certain 2×2 matrix A has columns $\mathbf{v}$ and $\mathbf{w}$, pictured below. Solve the equations $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the vectors in the picture.



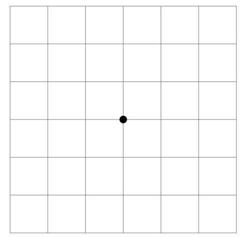
95. Compute the area of the triangle in the picture:



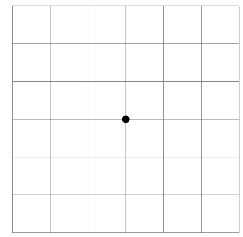
96. Give examples of 2×2 matrices A, B, C with ranks $0, 1$, and 2, respectively, and draw pictures of the null space and column space. (Be precise!)

a) Rank 0: $A = \begin{pmatrix} & \\ & \end{pmatrix}$

$N(A)$

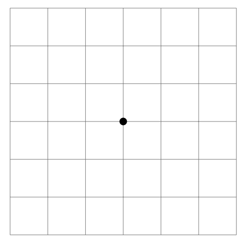


$C(A)$

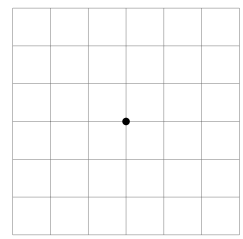


b) Rank 1: $B = \begin{pmatrix} & \\ & \end{pmatrix}$

$N(B)$

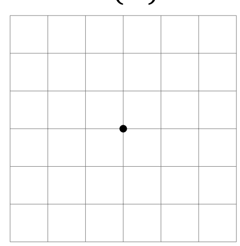


$C(B)$

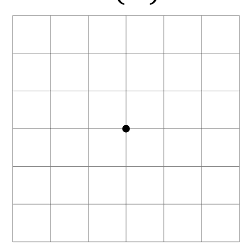


c) Rank 2: $C = \begin{pmatrix} & \\ & \end{pmatrix}$

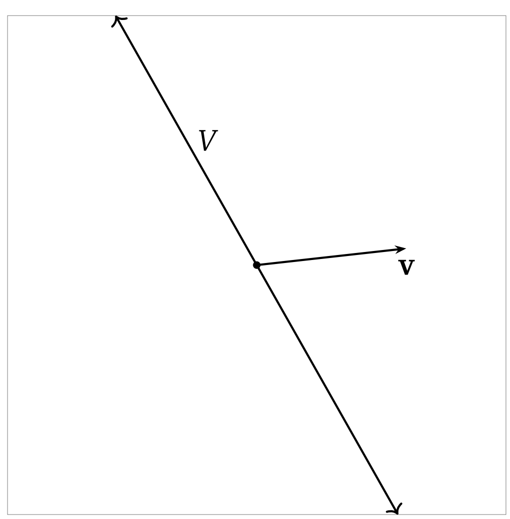
$N(C)$



$C(C)$

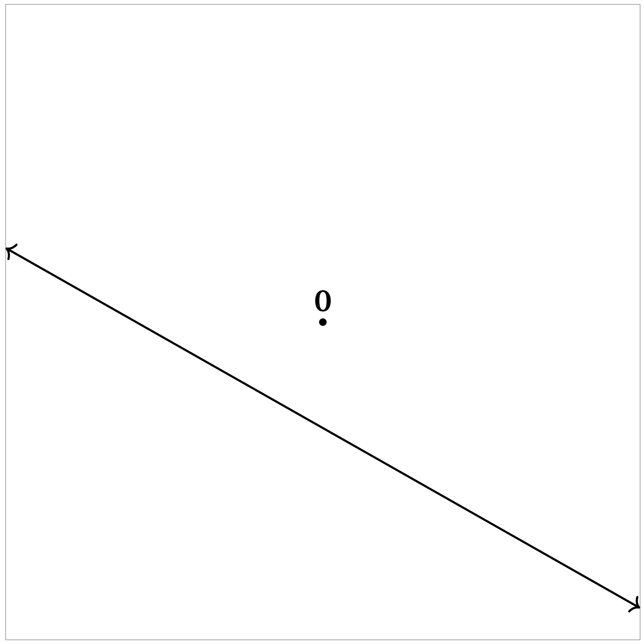


97. A subspace V and a vector $\mathbf{v}$ are drawn below. Draw the projection $\mathbf{p}$ of $\mathbf{v}$ onto V, and draw the projection $\mathbf{p}_\perp$ of $\mathbf{v}$ onto $V^\perp$. Label your answers!



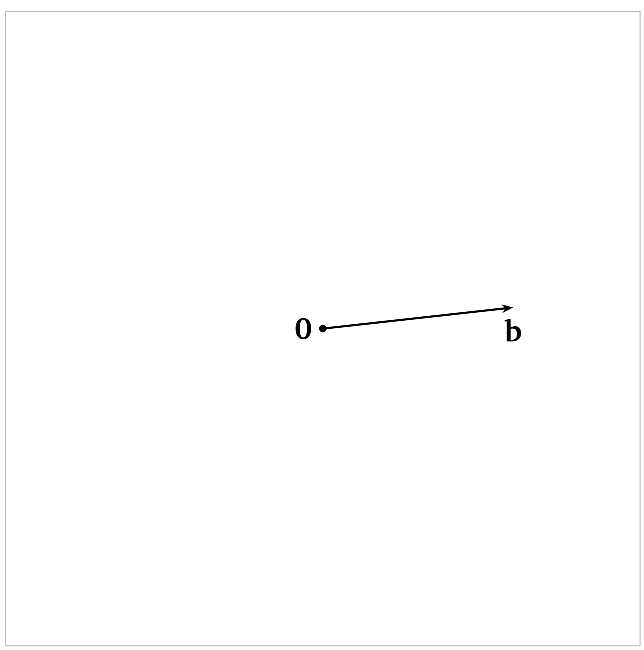
98. This problem concerns a certain 2×2 matrix A and a vector $\mathbf{b} \in \mathbf{R}^2$. You do not know what they are numerically.

a) The solutions of $A\mathbf{x} = \mathbf{b}$ are drawn below. Draw $N(A)$ in the same diagram.



b) The rank of A is ☐.

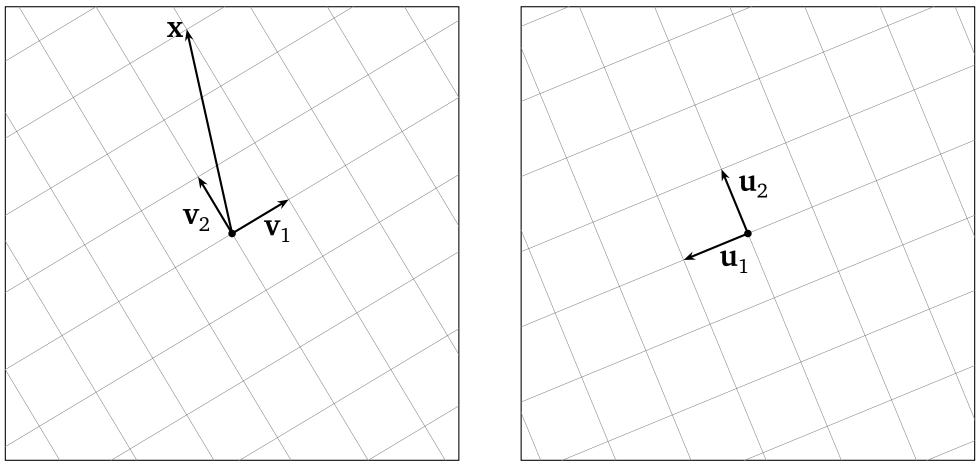
c) Suppose that $\mathbf{b}$ is the vector in the picture. Draw the left null space of A in the same picture. [This is the same $\mathbf{b}$ as before, so in particular, $A\mathbf{x} = \mathbf{b}$ has a solution.]



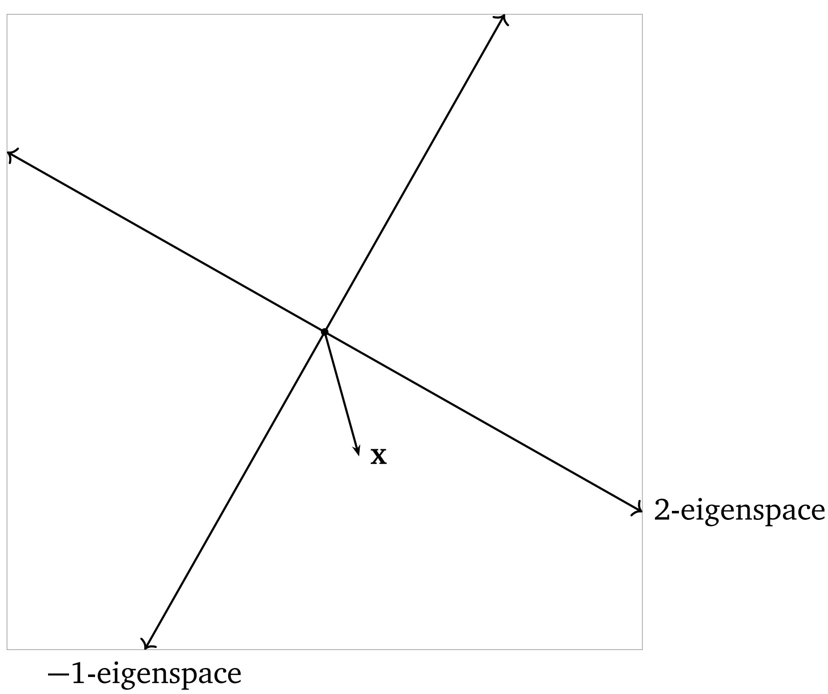
99. A certain 2×2 matrix A has the singular value decomposition

$$A = \begin{pmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix}^T,$$

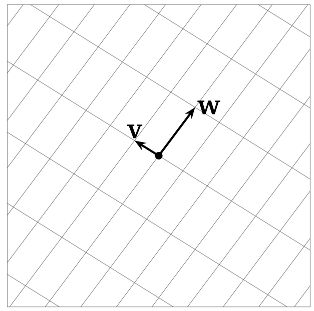
where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are drawn in the diagrams below. Given $\mathbf{x}$ in the diagram on the left, draw $A\mathbf{x}$ on the diagram on the right.



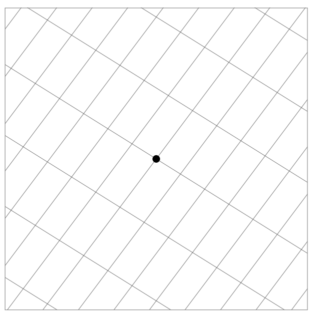
100. A certain 2×2 matrix A has eigenvalues 2 and -1, with eigenspaces drawn below. If $\mathbf{x}$ is the vector in the picture, draw $A\mathbf{x}$.



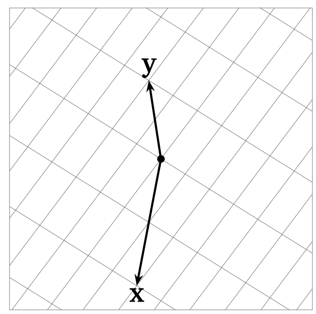
101. A certain 2×2 matrix A has eigenectors $\mathbf{v}$ and $\mathbf{w}$, pictured below, with corresponding eigenvalues $3/2$ and $-1/2$.



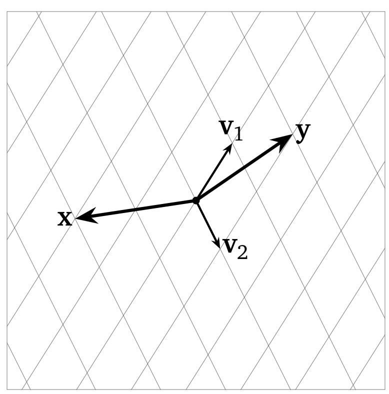
a) [3 points] Draw $A\mathbf{v}$ and $A\mathbf{w}$ below.



b) [4 points] Draw $A\mathbf{x}$ and $A\mathbf{y}$ below.



102. Suppose that $A = X\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}X^{-1}$, where X has columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Given $\mathbf{x}$ and $\mathbf{y}$ in the picture below, draw the vectors $A\mathbf{x}$ and $A\mathbf{y}$.



103. Find the 2×2 matrix A whose eigenspaces are drawn below. The grid lines are one unit apart.