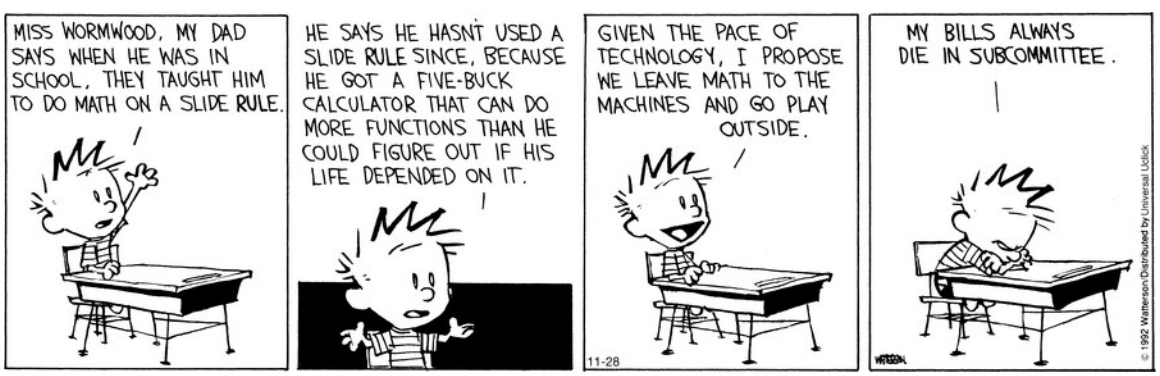
MATH 218 SECTION 3
MIDTERM EXAMINATION 1

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic, but you should not need one. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1.

[15 points]

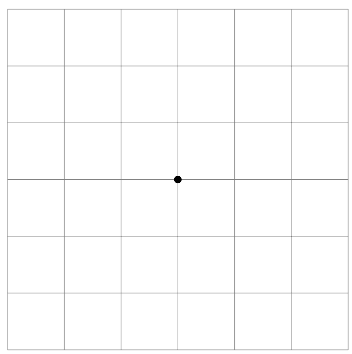
Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Use Gauss–Jordan elimination on A to put A into reduced row echelon form. Circle the free columns.

b) The rank of A is $\boxed{}$.

c) Draw a picture of the column space $C(A)$ below.



d) Find a spanning set for the null space of A.

$$N(A) = S\left\{ \qquad\qquad\qquad \right\}$$

e) The null space is a (circle one) $\begin{pmatrix} \text{point} \\ \text{line} \\ \text{plane} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{}}$.

Solution.

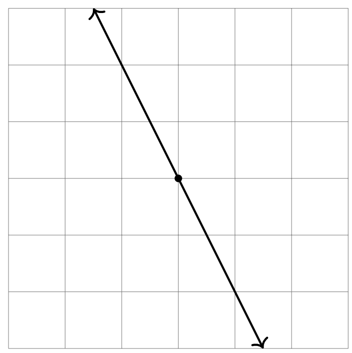
a) Only one row operation is needed:

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \xrightarrow{R_2 \mathrel{+}= 2R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The second and third columns are free.

b) There is one pivot, so the rank is one.

c) The column space is spanned by $\binom{1}{-2}$, as the other two columns are multiples of the first.



d) We write the reduced row echelon form as a system of equations:

$$\begin{aligned} x - y + 2z &= 0 \\ 0 &= 0. \end{aligned}$$

We ignore the second equation, move the free variables to the right-hand side, and columnate:

$$\begin{aligned} x &= y - 2z \\ y &= y \\ z &= \quad\;\; z \end{aligned} \qquad \xrightarrow{\text{vector equation}} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

This shows that

$$N(A) = S \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

e) The null space is a plane in $\mathbf{R}^3$.

Problem 2.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 5 & 4 & 1 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

a) Find the inverse of A.

$$A^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Express A^{-1} as a product of elementary matrices.

$$A^{-1} =$$

c) Solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is an unknown vector.

(Your answer will be a formula in b_1, b_2, b_3.)

$$\mathbf{x} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

Solution.

a) We augment by the identity matrix, then row reduce:

$$\left(\begin{array}{rrr|rrr} 5 & 4 & 1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{rrr|rrr} -1 & -2 & -1 & 0 & 1 & 0 \\ 5 & 4 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \times= -1} \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 0 & -1 & 0 \\ 5 & 4 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 -= 5R_1} \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & -6 & -4 & 1 & 5 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & -6 & -4 & 1 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \times= -1} \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & -6 & -4 & 1 & 5 & 0 \end{array}\right) \xrightarrow{R_3 += 6R_2} \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 5 & -6 \end{array}\right)$$

$$\xrightarrow{R_3 \div= 2} \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1/2 & 5/2 & -3 \end{array}\right) \xrightarrow{R_1 -= 2R_2} \left(\begin{array}{rrr|rrr} 1 & 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1/2 & 5/2 & -3 \end{array}\right)$$

$$\xrightarrow{R_2 -= R_3} \left(\begin{array}{rrr|rrr} 1 & 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1/2 & -5/2 & 2 \\ 0 & 0 & 1 & 1/2 & 5/2 & -3 \end{array}\right) \xrightarrow{R_1 += R_3} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/2 & 3/2 & -1 \\ 0 & 1 & 0 & -1/2 & -5/2 & 2 \\ 0 & 0 & 1 & 1/2 & 5/2 & -3 \end{array}\right)$$

Since the left half of this matrix is the identity, the right is the inverse:

$$A^{-1} = \begin{pmatrix} 1/2 & 3/2 & -1 \\ -1/2 & -5/2 & 2 \\ 1/2 & 5/2 & -3 \end{pmatrix}.$$

b) The inverse matrix is the product of the elementary matrices for the row operations we did, with the first operation on the left:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) $$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1/2 & 3/2 & -1 \\ -1/2 & -5/2 & 2 \\ 1/2 & 5/2 & -3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}b_1 + \frac{3}{2}b_2 - b_3 \\ -\frac{1}{2}b_1 - \frac{5}{2}b_2 + 2b_3 \\ \frac{1}{2}b_1 + \frac{5}{2}b_2 - 3b_3 \end{pmatrix}.$$

Problem 3. [10 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ -4 & -5 & -3 \\ -2 & -6 & 0 \end{pmatrix}.$$

a) Find a lower-triangular matrix L with ones on the diagonal and an upper-triangular matrix U such that $A = LU$.

$$L = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad U = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Solve the equation $A\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ using the LU decomposition you found above.

$$\mathbf{x} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

Solution.

a) First we row reduce using only row replacements:

$$\begin{pmatrix} 2 & 3 & 1 \\ -4 & -5 & -3 \\ -2 & -6 & 0 \end{pmatrix} \xrightarrow{R_2 \mathrel{+}= 2R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & -6 & 0 \end{pmatrix} \xrightarrow{R_3 \mathrel{+}= R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & 1 \end{pmatrix} \xrightarrow{R_3 \mathrel{+}= 3R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

The last matrix is U. To find L, we undo each row operation, in the reverse order:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \mathrel{-}= 3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \xrightarrow{R_3 \mathrel{-}= R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \xrightarrow{R_2 \mathrel{-}= 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

The last matrix is L. To summarize:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

b) First we solve $L\mathbf{c} = \mathbf{b}$ using forward substitution:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} &\quad \Longrightarrow c_1 = 0 \\ -2c_2 + c_1 = 0 &\quad \Longrightarrow c_2 = 0 \\ -c_1 - 3c_2 + c_3 = 4 &\quad \Longrightarrow c_3 = 4. \end{aligned}$$

Now we solve $U\mathbf{x} = \mathbf{c}$ by back substitution:

$$\begin{aligned} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} &\quad \Longrightarrow x_3 = -2 \\ x_2 - x_3 = 0 &\quad \Longrightarrow x_2 = -2 \\ 2x_1 + 3x_2 + x_3 = 0 &\quad \Longrightarrow x_1 = 4. \end{aligned}$$

The solution is

$$\mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}.$$

Problem 4.

[10 points]

a) Briefly explain why a system of linear equations $A\mathbf{x} = \mathbf{0}$ cannot have exactly two solutions.

b) Give an example of a **non-invertible** 3×3 matrix A with **no nonzero entries**.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

c) Consider the subspace

$$V = S\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}.$$

Find two other vectors that span V. (You may not include scalar multiples of the vectors $\binom{1}{1}$ or $\binom{1}{-1}$.)

$$V = S\left\{\begin{pmatrix} \\ \end{pmatrix}, \begin{pmatrix} \\ \end{pmatrix}\right\}.$$

Solution.

a) The solutions of $A\mathbf{x} = \mathbf{0}$ form the null space $N(A)$, which is a subspace. Any subspace contains the zero vector; if $N(A)$ contains another (a non-zero) vector $\mathbf{x}$, then it contains the infinitely many scalar multiples of $\mathbf{x}$ as well, which are also solutions of $A\mathbf{x} = \mathbf{0}$.

b) There are many examples; here is one:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

This matrix only has one pivot, so it is not invertible.

c) Since $\binom{1}{1}$ and $\binom{1}{-1}$ are not scalar multiples of each other, the subspace V is all of $\mathbf{R}^2$. This is spanned by any two nonzero, noncollinear vectors; for example,

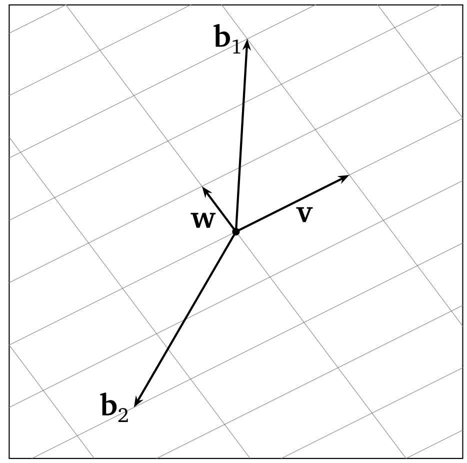
$$V = S\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

Problem 5. [10 points]

A certain 2×2 matrix A has columns $\mathbf{v}$ and $\mathbf{w}$, pictured below. Solve the equations $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the vectors in the picture.

a) $\mathbf{x}_1 =$

b) $\mathbf{x}_2 =$



Solution.

If $A\mathbf{x} = \mathbf{b}$, then the entries of $\mathbf{x}$ are the coefficients of a linear combination of the columns of A: that is, $x_1\mathbf{v} + x_2\mathbf{w} = \mathbf{b}$. We can see from the picture which coefficients are needed to reach $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$\mathbf{b}_1 = \mathbf{v} + 3\mathbf{w} \implies \mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \mathbf{b}_2 = -\frac{3}{2}\mathbf{v} - 2\mathbf{w} \implies \mathbf{x}_2 = \begin{pmatrix} -3/2 \\ -2 \end{pmatrix}.$$