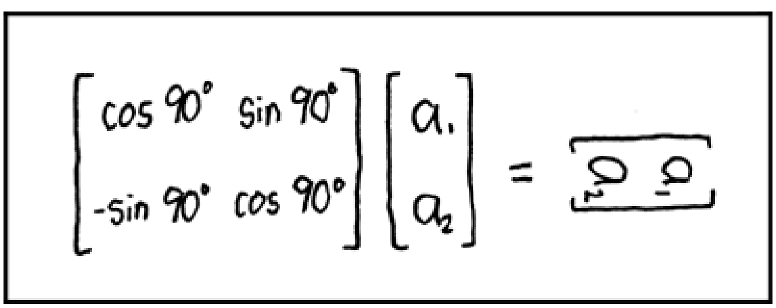
MATH 218 SECTION 3
MIDTERM EXAMINATION 2

Name		Duke ID Number	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



[Hint: this is a joke.]

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Problem 1. [20 points]

Consider the matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 1 & 7 & 5 \\ -2 & -2 & 4 & 8 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

a) Compute a basis for the null space of A: $\left\{ \qquad \right\}$

b) Compute a basis for the row space of A: $\left\{ \qquad \right\}$

c) Find the complete solution of $A\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. (Note that a particular solution is $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.)

d) The dimension of $C(A)$ is $\boxed{}$

e) Find a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has no solution, or explain why no such $\mathbf{b}$ exists.

Solution.

a) We write the reduced row echelon form as a system of equations:

$$\begin{aligned} x_1 + x_2 \quad - 2x_4 &= 0 \\ x_3 + \ x_4 &= 0 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x_1 &= -x_2 + 2x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Hence $N(A)$ has basis $\{(-1,1,0,0), (2,0,-1,1)\}$.

b) The rows of the reduced row echelon form of A are not multiples of each other, so they form a basis for the row space: $\{(1,1,0,-2), (0,0,1,1)\}$.

c) A general solution of $A\mathbf{x} = \binom{1}{-2}$ is a particular solution plus a vector in the null space:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

d) We see from the reduced row echelon form that A has two pivots, so $C(A)$ has dimension 2.

e) This is not possible, since A has full row rank.

Problem 2. [20 points]

Consider the subspace $V = \left\{ \text{all solutions of } \begin{array}{r} x_1 + x_2 + 7x_3 + 5x_4 = 0 \\ -2x_1 - 2x_2 + 4x_3 + 8x_4 = 0 \end{array} \right\}$.

a) Find an orthonormal basis of V: $\left\{ \qquad \right\}$

b) Find an orthonormal basis of $V^\perp$: $\left\{ \qquad \right\}$

c) Compute the matrix P for projection onto V: $P = \begin{pmatrix} \qquad \end{pmatrix}$

d) Compute the projection $\mathbf{p}$ of $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ onto $V^\perp$: $\mathbf{p} = \begin{pmatrix} \quad \end{pmatrix}$

Solution.

Note that V is the null space of the matrix A from Problem 1.

a) We apply Gram–Schmidt to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ from Problem 1(a):

$$\mathbf{p}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{p}_2 = \mathbf{v}_2 - \frac{\mathbf{p}_1 \cdot \mathbf{v}_2}{\mathbf{p}_1 \cdot \mathbf{p}_1}\mathbf{p}_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} - \frac{-2}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Dividing by the lengths, we obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

b) We apply Gram–Schmidt to the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ from Problem 1(b):

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix} \qquad \mathbf{p}_2 = \mathbf{w}_2 - \frac{\mathbf{p}_1 \cdot \mathbf{w}_2}{\mathbf{p}_1 \cdot \mathbf{p}_1}\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{-2}{6}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

Dividing by the lengths, we obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

c) The projection matrix is QQ^T, where Q is the orthogonal matrix whose columns are the orthonormal basis vectors in (a):

$$P = QQ^T = \begin{pmatrix} -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 1/2 \\ 0 & -1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

d) The projection onto V is

$$\frac{1}{4}\begin{pmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ 0 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ 1/2 \end{pmatrix}.$$

Hence the projection onto $V^\perp$ is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Problem 3. [10 points]

All of the following statements are false. Provide a counterexample to each. You need not justify your answers.

a) Any two vectors in $\mathbf{R}^3$ span a plane.

b) If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbf{R}^m$.

c) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^\perp$.

d) Any 2×2 orthogonal matrix has the form

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Solution.

a) The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$ span a line, not a plane.

b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ does not have a solution for $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

c) Take V to be the x-axis in $\mathbf{R}^3$, and W to be the y-axis.

d) $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is orthogonal.

Problem 4. [20 points]

a) Give an example of a 3×2 matrix A and a vector $\mathbf{b}$ in $\mathbf{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has more than one least-squares solution.

b) Give an example of a 3×3 matrix A such that $C(A) = N(A)$, or explain why no such matrix exists.

c) Write three different nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbf{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly *de*pendent, but $\mathbf{v}_3$ is *not* in $S\{\mathbf{v}_1, \mathbf{v}_2\}$. Clearly indicate which is $\mathbf{v}_3$.

d) Explain why every projection matrix P can be written as QQ^T for an orthogonal matrix Q.

Solution.

a) Any matrix with (zero or) one pivot works for any vector $\mathbf{b}$. For instance,

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

b) The rank theorem states that $\dim C(A) + \dim N(A) = 3$. Since 3 is an odd number, it cannot be equal to $2\dim C(A)$, so $N(A) = C(A)$ is impossible.

c) For instance, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$

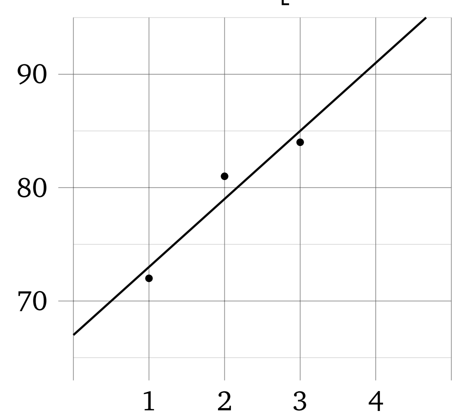
d) Suppose that P is the projection onto a subspace V. Choose an orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_n$ for V—this is possible by Gram–Schmidt. Then $P = QQ^T$, where Q has columns $\mathbf{q}_1, \ldots, \mathbf{q}_n$.

Problem 5. [10 points]

Your roomate Karxon is currently taking Math 105L. Karxon scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Karxon does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) The general equation of a line in $\mathbf{R}^2$ is $y = Cx + D$. Write down the system of linear equations in C and D that would be satisfied by a line passing through the points $(1, 72)$, $(2, 81)$, and $(3, 84)$, and then write down the corresponding matrix equation.

b) Solve the corresponding least squares problem for C and D, and use this to *write down* and *draw* the the best fit line below. [Use a calculator]



$y = 67 + 6x$

c) What score does this line predict for the fourth (final) exam?

Solution.

a)

$$\begin{aligned} 1C + D &= 72 \\ 2C + D &= 81 \\ 3C + D &= 84 \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix}$$

b) The normal equation is $A^T A\mathbf{x} = A^T\mathbf{b}$, which gives rise to the following augmented matrix:

$$\left(\begin{array}{cc|c} 14 & 6 & 486 \\ 6 & 3 & 237 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 67 \end{array}\right)$$

The equation and picture are on the previous page.

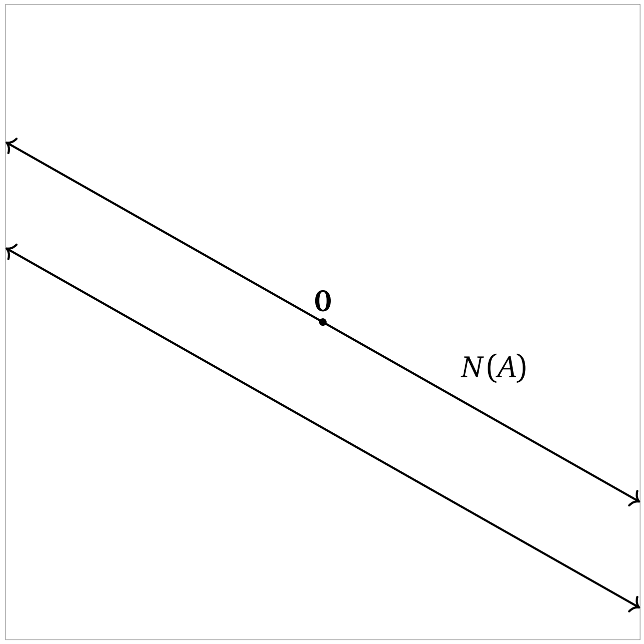
c) $67 + 6 \cdot 4 = 91$

Problem 6.

[10 points]

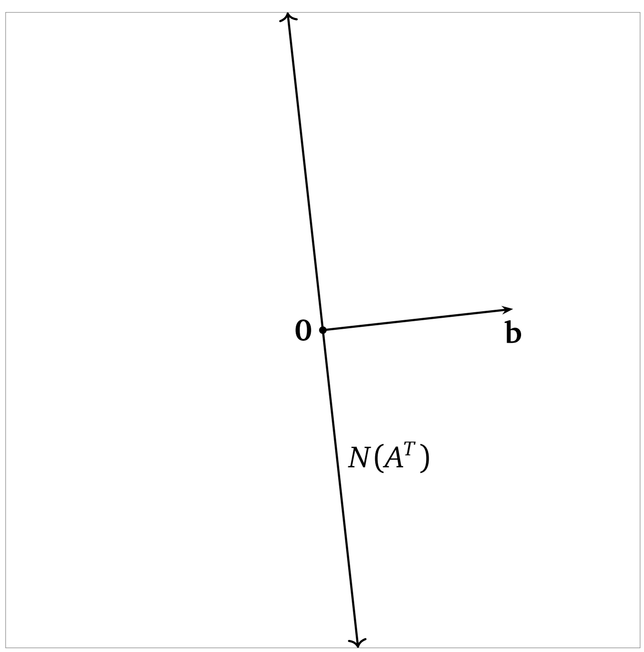
This problem concerns a certain 2×2 matrix A and a vector $\mathbf{b} \in \mathbf{R}^2$. You do not know what they are numerically.

a) The solutions of $A\mathbf{x} = \mathbf{b}$ are drawn below. Draw $N(A)$ in the same diagram.



b) The rank of A is $\boxed{1}$.

c) Suppose that $\mathbf{b}$ is the vector in the picture. Draw the left null space of A in the same picture. [This is the same $\mathbf{b}$ as before, so in particular, $A\mathbf{x} = \mathbf{b}$ has a solution.]



Solution.

a) The null space is the parallel line through the origin. See the previous page for the picture.

b) We have $\dim N(A) + \dim C(A) = 1 + \dim C(A) = 2$, so $C(A)$ has dimension 1, and hence the rank of A is 1.

c) The column space is a line, and it contains $\mathbf{b}$ because $A\mathbf{x} = \mathbf{b}$ is consistent. Hence $C(A)$ is the line through $\mathbf{b}$, and $N(A^T)$ is the orthogonal complement.