MATH 218 SECTION 3
MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

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Problem 1.

[20 points]

Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

From this you can read off all of the following properties of A.

a) A is a $\square \times \square$ matrix of rank $r = \square$.

b) Find orthonormal bases of the four fundamental subspaces of A:

$N(A)$	$C(A^T)$	$C(A)$	$N(A^T)$
$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$

c) Express A as a linear combination of rank-one matrices $\mathbf{u}\mathbf{v}^T$ (your answer should consist of vectors with numbers, not letters):

$A =$

d) Find a *unit* vector $\mathbf{x}$ maximizing $|A\mathbf{x}|$:

$\mathbf{x} = \begin{pmatrix} \\ \\ \end{pmatrix}$ $|A\mathbf{x}| = \square$

e) Compute the matrix P for orthogonal projection onto $C(A)$ (write it as a product, without expanding it out):

$P =$

Solution.

a) The size and rank of A can be read off from Σ: A is a 4×3 matrix of rank 2.

b) The columns of U and V give orthonormal bases for the four subspaces.

$$N(A): \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$C(A^T): \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

$$C(A): \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \right\}$$

$$N(A^T): \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

c) This is the vector form of the SVD:

$$A = 3 \cdot \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} + 2 \cdot \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & -2 & -1 \end{pmatrix}$$

d) The maximum value of $|A\mathbf{x}|$ is σ_1 and is attained at $\mathbf{v}_1$:

$$\mathbf{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad |A\mathbf{x}| = 3$$

e) The orthogonal projection onto $C(A)$ is $AA^+ = (U\Sigma V^T)(V\Sigma^+ U^T) = U\Sigma\Sigma^+ U^T$, which is

$$P = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^T.$$

Problem 2.

[20 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

a) Find an orthogonal matrix Q and a diagonal matrix Λ such that $S = Q\Lambda Q^T$:

$$Q = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad \Lambda = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Circle one: S is

positive-definite   positive-semidefinite   neither of these

c) Write down the singular value decomposition of S:

$$S = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Solution.

a) First we compute the eigenvalues of S. We find the characteristic polynomial by expanding cofactors along the first column:

$$\begin{aligned}
p(\lambda) = \det\begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{pmatrix} \\
&= (1-\lambda)\det\begin{pmatrix} -1-\lambda & -2 \\ -2 & -\lambda \end{pmatrix} + 2\det\begin{pmatrix} 0 & 2 \\ -1-\lambda & -2 \end{pmatrix} \\
&= (1-\lambda)\big[(-1-\lambda)(-\lambda)-4\big] + 2\big[-2(-1-\lambda)\big] \\
&= (1-\lambda)(\lambda^2+\lambda-4) - 4(-1-\lambda) \\
&= \lambda^2+\lambda-4-\lambda^3-\lambda^2+4\lambda+4+4\lambda \\
&= -\lambda^3+9\lambda = -\lambda(\lambda-3)(\lambda+3).
\end{aligned}$$

The eigenvalues are 0 and ± 3; we compute eigenvectors:

$$\lambda = 3: \quad S-3I = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 2 \\ 2 & -2 & -3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \mathbf{v}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\lambda = -3: \quad S+3I = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & -2 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \mathbf{v}_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda = 0: \quad S-0I = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \mathbf{v}_3 = \frac{1}{3}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

Hence $S = Q\Lambda Q^T$ for

$$Q = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) S has a negative eigenvalue, so it is neither positive-definite nor positive-semidefinite.

c) The singular values of S are the absolute values of the nonzero eigenvalues: $\sigma_1 = \sigma_2 = 3$. We have

$$\begin{aligned}
3\mathbf{v}_1 = S\mathbf{v}_1 = \sigma_1\mathbf{u}_1 &\implies \mathbf{u}_1 = \mathbf{v}_1 \\
-3\mathbf{v}_2 = S\mathbf{v}_2 = \sigma_2\mathbf{u}_2 &\implies \mathbf{u}_2 = -\mathbf{v}_2.
\end{aligned}$$

We can take $\mathbf{u}_3 = \mathbf{v}_3$ as our orthonormal basis of $N(S) = N(S^T)$, so

$$S = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -1 & -2 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ -2 & -2 & 1 \end{pmatrix}$$

Problem 3. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

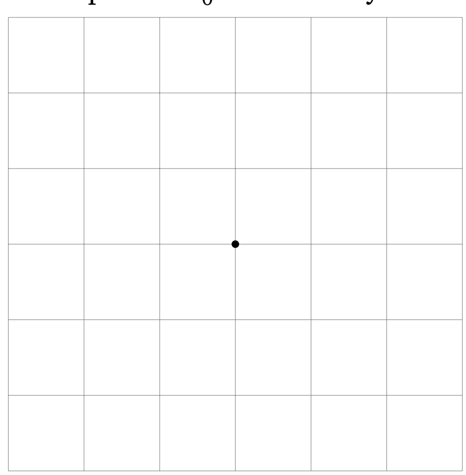
a) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

$$X = \begin{pmatrix} & \\ & \end{pmatrix} \qquad \Lambda = \begin{pmatrix} & \\ & \end{pmatrix}$$

b) Compute $A^n\mathbf{v}_0$ for $\mathbf{v}_0 = \binom{1}{2}$. What happens when $n \to \infty$?

$$A^n\mathbf{v}_0 = \begin{pmatrix} \\ \end{pmatrix} \qquad A^n\mathbf{v}_0 \xrightarrow{n\to\infty} \begin{pmatrix} \\ \end{pmatrix}$$

c) In the diagram, *draw and label* the eigenspaces of A, and draw the vectors $\mathbf{v}_0, A\mathbf{v}_0, A^2\mathbf{v}_0, A^3\mathbf{v}_0, \ldots$ as points. (The grid lines are one unit apart, and the dot is the origin.) [Hint: you do not have to compute $A^n\mathbf{v}_0$ numerically to do this.]



d) Solve the system of ordinary differential equations

$$\begin{aligned} \tfrac{d}{dt}u_1 &= 2u_1 - u_2 & u_1(0) &= 1 \\ \tfrac{d}{dt}u_2 &= \tfrac{3}{2}u_1 - \tfrac{1}{2}u_2 & u_2(0) &= 2 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} u_1(t) &= \\ u_2(t) &= \end{aligned}$$

Solution.

a) The characteristic polynomial of A is

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$. We compute eigenvectors:

$$A - 1I = \begin{pmatrix} 1 & -1 \\ - & - \end{pmatrix} \rightsquigarrow \mathbf{q}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A - \frac{1}{2}I = \begin{pmatrix} \frac{3}{2} & -1 \\ - & - \end{pmatrix} \rightsquigarrow \mathbf{q}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence $A = X\Lambda X^{-1}$ for

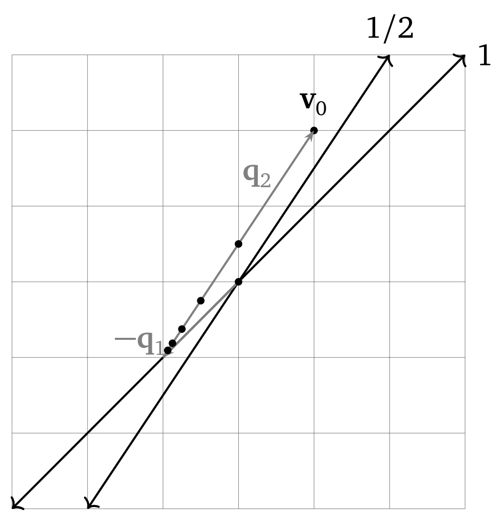
$$X = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

b) We eyeball $\mathbf{v}_0 = -\mathbf{q}_1 + \mathbf{q}_2$, so

$$A^n \mathbf{v}_0 = -\mathbf{q}_1 + \frac{1}{2^n}\mathbf{q}_2.$$

This approaches $-\mathbf{q}_1 = \binom{-1}{-1}$ as $n \to \infty$.

c) The eigenspaces are spanned by $\mathbf{q}_1$ and $\mathbf{q}_2$.



d) We need to solve $\mathbf{u}' = A\mathbf{u}$ for $\mathbf{u}(0) = \binom{1}{2} = \mathbf{v}_0$ and A as above. We have $\mathbf{u}(0) = -\mathbf{q}_1 + \mathbf{q}_2$, so the solution is

$$\mathbf{u}(t) = -e^t\mathbf{q}_1 + e^{t/2}\mathbf{q}_2 \quad \Longrightarrow \quad \begin{aligned} u_1 &= -e^t + 2e^{t/2} \\ u_2 &= -e^t + 3e^{t/2} \end{aligned}$$

Problem 4. [16 points]

All of the following statements are false. Provide a counterexample to each. You need not justify your answers.

a) The singular values of a diagonalizable, invertible, square matrix are the absolute values of the eigenvalues. [Hint: try a 2×2 matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$.]

b) If a matrix has determinant zero, then two of the columns are multiples of each other, or one of the columns is zero.

c) If S is symmetric, then either S or $-S$ is positive-semidefinite.

d) If λ is an eigenvalue of AA^T, then λ is an eigenvalue of A^TA.

Solution.

a) The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 1 and 2, so it is invertible and diagonalizable. However,

$$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 4$; we have $p(1^2) = -1$ and $p(2^2) = -4$, so neither 1 nor 2 is a singular value of A.

b) One column can be the sum of the other two; for instance,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix}.$$

c) Any symmetric matrix S with both positive and negative eigenvalues works:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

d) This can only be true for $\lambda = 0$ (see 5(a)). Zero is an eigenvalue of $A^T A$ (resp. AA^T) if and only if A has linearly independent columns (resp. rows), so we need to find a matrix with linearly independent columns and linearly dependent rows. For instance:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 5. [16 points]

Prove the following statements. (None should take more than a couple of lines.)

a) If λ is a eigenvalue of $A^T A$ and $\lambda > 0$, then λ is also an eigenvalue of AA^T.

b) If A is a 3×3 matrix that has eigenvalues 1 and -1, both of algebraic multiplicity one, then A is diagonalizable (over the real numbers).

c) The row space of A equals the row space of $A^T A$.

d) If A has linearly independent columns then $A^+ A$ is the identity matrix.

Solution.

a) Say $A^TA\mathbf{v} = \lambda\mathbf{v}$. Set $\mathbf{u} = A\mathbf{v}$; this is not zero because $A^T\mathbf{u} = A^TA\mathbf{v} = \lambda\mathbf{v} \neq 0$. Then

$$AA^T\mathbf{u} = AA^TA\mathbf{v} = A(A^TA\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda\mathbf{u},$$

so $\mathbf{u}$ is an eigenvector of AA^T with eigenvalue λ.

b) The characteristic polynomial $p(\lambda)$ has roots at ± 1. Since $p(\lambda)$ has degree 3, it has a third root λ; since complex roots of p come in conjugate pairs, λ must be real. Since ± 1 have algebraic multiplicity one, $\lambda \neq \pm 1$, so A has three distinct eigenvalues, hence is diagonalizable.

c) The null space of A^TA is equal to $N(A)$. The row space of A^TA is the orthogonal complement of $N(A^TA) = N(A)$, as is the row space of A^TA.

d) Since A has linearly independent columns, its null space is zero, so its row space is all of $\mathbf{R}^n$. But A^+A is the projection onto the row space of A, and the projection onto all of $\mathbf{R}^n$ is the identity.

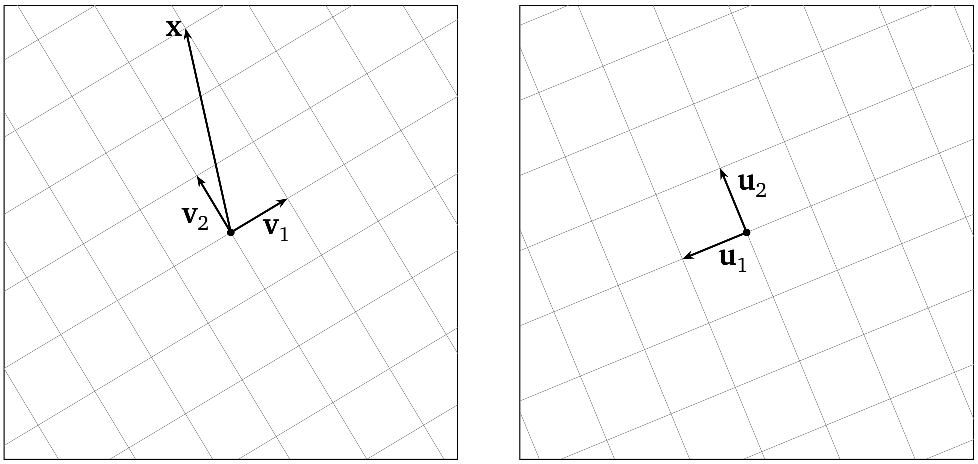
Problem 6.

[8 points]

A certain 2×2 matrix A has the singular value decomposition

$$A = \begin{pmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix}^T,$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are drawn in the diagrams below. Given $\mathbf{x}$ in the diagram on the left, draw $A\mathbf{x}$ on the diagram on the right.



Solution.

We have $\mathbf{x} = \mathbf{v}_1 + 3\mathbf{v}_2$. The way the SVD works, we have $A\mathbf{v}_1 = 2\mathbf{u}_1$ and $A\mathbf{v}_2 = 0$, so

$$A\mathbf{x} = A\mathbf{v}_1 + 3A\mathbf{v}_2 = 2\mathbf{u}_1.$$