MATH 218 SECTION 3
MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

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Problem 1. [20 points]

Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

From this you can read off all of the following properties of A.

a) A is a $\square \times \square$ matrix of rank $r = \square$.

b) Find orthonormal bases of the four fundamental subspaces of A:

$N(A)$	$C(A^T)$	$C(A)$	$N(A^T)$
$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$

c) Express A as a linear combination of rank-one matrices $\mathbf{u}\mathbf{v}^T$ (your answer should consist of vectors with numbers, not letters):

$A =$

d) Find a *unit* vector $\mathbf{x}$ maximizing $|A\mathbf{x}|$:

$\mathbf{x} = \begin{pmatrix} \\ \\ \end{pmatrix}$ $\quad |A\mathbf{x}| = \square$

e) Compute the matrix P for orthogonal projection onto $C(A)$ (write it as a product, without expanding it out):

$P =$

[Scratch work for problem 1]

Problem 2.

[20 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

a) Find an orthogonal matrix Q and a diagonal matrix Λ such that $S = Q\Lambda Q^T$:

$$Q = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

b) Circle one: S is

positive-definite   positive-semidefinite   neither of these

c) Write down the singular value decomposition of S:

$$S = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

[Scratch work for problem 2]

Problem 3.

[20 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

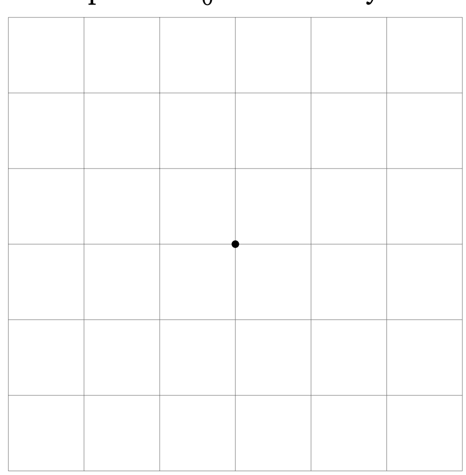
a) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

$$X = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

b) Compute $A^n\mathbf{v}_0$ for $\mathbf{v}_0 = \binom{1}{2}$. What happens when $n \to \infty$?

$$A^n\mathbf{v}_0 = \begin{pmatrix} \quad \\ \quad \end{pmatrix} \qquad A^n\mathbf{v}_0 \xrightarrow{n\to\infty} \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

c) In the diagram, *draw and label* the eigenspaces of A, and draw the vectors $\mathbf{v}_0, A\mathbf{v}_0, A^2\mathbf{v}_0, A^3\mathbf{v}_0, \ldots$ as points. (The grid lines are one unit apart, and the dot is the origin.) [Hint: you do not have to compute $A^n\mathbf{v}_0$ numerically to do this.]



d) Solve the system of ordinary differential equations

$$\begin{aligned} \tfrac{d}{dt}u_1 &= 2u_1 - u_2 & u_1(0) &= 1 \\ \tfrac{d}{dt}u_2 &= \tfrac{3}{2}u_1 - \tfrac{1}{2}u_2 & u_2(0) &= 2 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} u_1(t) &= \\ u_2(t) &= \end{aligned}$$

[Scratch work for problem 3]

Problem 4. [16 points]

All of the following statements are false. Provide a counterexample to each. You need not justify your answers.

a) The singular values of a diagonalizable, invertible, square matrix are the absolute values of the eigenvalues. [Hint: try a 2×2 matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$.]

b) If a matrix has determinant zero, then two of the columns are multiples of each other, or one of the columns is zero.

c) If S is symmetric, then either S or $-S$ is positive-semidefinite.

d) If λ is an eigenvalue of AA^T, then λ is an eigenvalue of A^TA.

[Scratch work for problem 4]

Problem 5. [16 points]

Prove the following statements. (None should take more than a couple of lines.)

a) If λ is a eigenvalue of $A^T A$ and $\lambda > 0$, then λ is also an eigenvalue of AA^T.

b) If A is a 3×3 matrix that has eigenvalues 1 and -1, both of algebraic multiplicity one, then A is diagonalizable (over the real numbers).

c) The row space of A equals the row space of $A^T A$.

d) If A has linearly independent columns then $A^+ A$ is the identity matrix.

[Scratch work for problem 5]

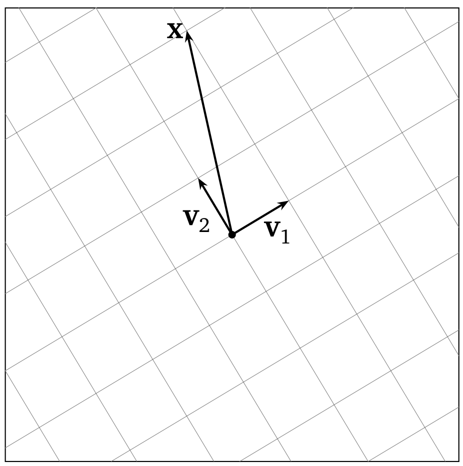
Problem 6.

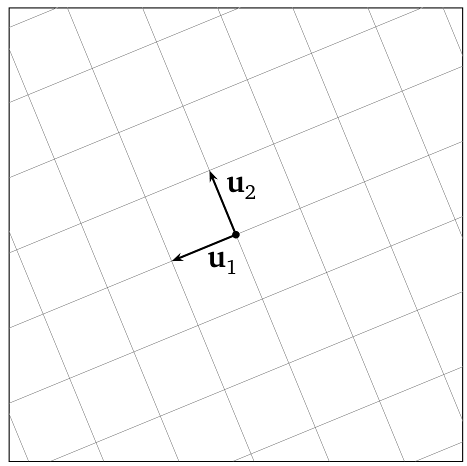
[8 points]

A certain 2×2 matrix A has the singular value decomposition

$$A = \begin{pmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix}^T,$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are drawn in the diagrams below. Given $\mathbf{x}$ in the diagram on the left, draw $A\mathbf{x}$ on the diagram on the right.





[Scratch work for problem 6]