MATH 218 SECTION 3
PRACTICE MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic, but you should not need one. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

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Problem 1. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 23 & 36 \\ 36 & 2 \end{pmatrix}.$$

a) Find the eigenvalues of this matrix. (One of the eigenvalues is -25.)

b) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

c) Consider the following system of differential equations with initial values:

$$\begin{aligned} \frac{dx}{dt} &= 23x + 36y \qquad & x(0) &= 7 \\ \frac{dy}{dt} &= 36x + 2y \qquad & y(0) &= -1. \end{aligned}$$

Find the solutions $x(t)$ and $y(t)$.

d) Find a formula for A^n in terms of n.

e) Find the vector $\mathbf{v} = \binom{x}{y}$ maximizing $|\mathbf{v}|$, subject to the constraint

$$23x^2 + 72xy + 2y^2 = 1.$$

[Scratch work for problem 1]

Problem 2. [10 points]

Let V be a subspace of $\mathbf{R}^4$; assume that $V \neq \{0\}$ and $V \neq \mathbf{R}^4$. Let P be the matrix for the projection onto V.

a) Show that the eigenvalues of P are 0 and 1.

b) Which of the four fundamental subspaces is equal to the 0-eigenspace? Which of the four fundamental subspaces is equal to the 1-eigenspace?

c) Why is P diagonalizable? What diagonal matrix is it similar to?

[Scratch work for problem 2]

Problem 3. [5 points]

Show that if S and T are positive definite, then so is $S+T$. [Hint: knowing that the eigenvalues are positive is not helpful here.]

Problem 4. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

This question is about the singular value decomposition $A = U\Sigma V^T$. Let $U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$ and $V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$.

a) Find $\mathbf{u}_1$ and $\mathbf{v}_1$ *without* computing A^TA or AA^T. [Hint: in which spaces do these live?]

b) Using your answers to (a) or otherwise, compute σ_1.

c) Write down the singular value decomposition $A = U\Sigma V^T$ of A.

[Scratch work for problems 3 and 4]

Problem 5.

[15 points]

Give examples of matrices with the following properties. If no such matrix exists, write "not possible". *All matrices in this problem have real entries.*

a) A 3×2 matrix with singular values 2 and 1.

b) A 2×2 matrix that is invertible but not diagonalizable.

c) A 3×3 symmetric matrix that is positive semidefinite but not positive definite.

d) A 3×3 symmetric matrix that is not diagonalizable.

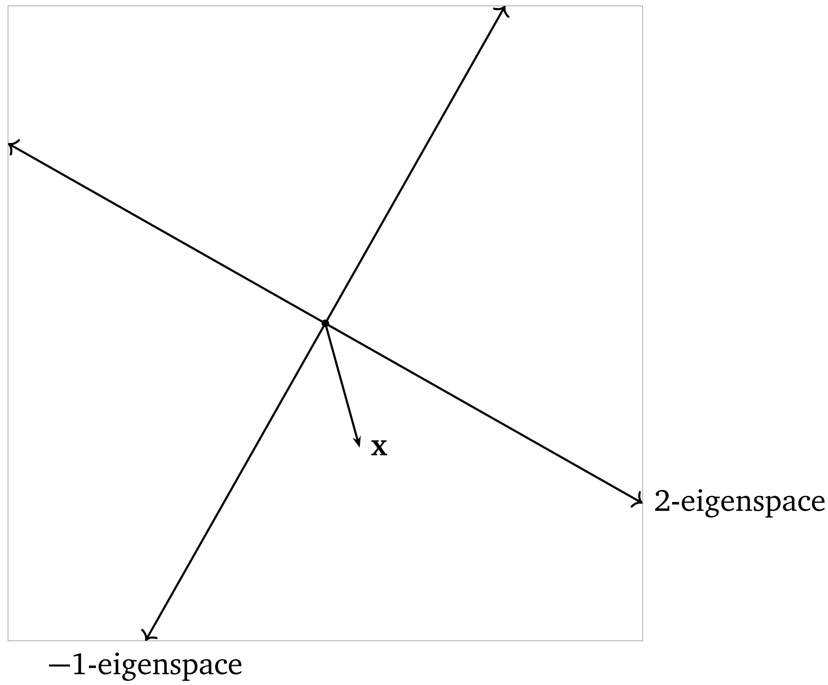
e) A 3×3 matrix whose entries are all either 1 or -1, with determinant 4.
[Hint: think in terms of maximizing volumes.]

[Scratch work for problem 5]

Problem 6.

[15 points]

A certain 2×2 matrix A has eigenvalues 2 and -1, with eigenspaces drawn below. If $\mathbf{x}$ is the vector in the picture, draw $A\mathbf{x}$.



[Scratch work for problem 6]