MATH 218D
PRACTICE FINAL EXAMINATION

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- You have 200 minutes to complete this exam and upload your work. The exam itself is meant to take 100 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear.
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ______________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ______________________________ Time: ______________

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.

Problem 1. [20 points]

Consider the sequence of numbers $0, 1, 5, 31, 185, \ldots$ given by the recursive formula

$$a_0 = 0 \qquad a_1 = 1 \qquad a_n = 5a_{n-1} + 6a_{n-2}.$$

a) Find a matrix A such that

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}.$$

b) Find the eigenvalues of A, and find corresponding eigenvectors.

c) Give a non-recursive formula for a_n.

Solution.

a) The matrix is $A = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix}$.

b) The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -1$, with corresponding eigenvectors $w_1 = \binom{1}{6}$ and $w_2 = \binom{-1}{1}$.

c) First we write $\binom{a_0}{a_1} = \binom{0}{1}$ in terms of our eigenbasis:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{7}(w_1 + w_2).$$

Hence we have

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{7}\big(6^n w_1 + (-1)^n w_2\big) = \frac{6^n}{7}\begin{pmatrix} 1 \\ 6 \end{pmatrix} + \frac{(-1)^n}{7}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The first coordinate is

$$a_n = \frac{1}{7}\big(6^n - (-1)^n\big).$$

Problem 2.

[20 points]

A certain matrix A has singular value decomposition $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{pmatrix}.$$

a) What is the rank of A?

b) What is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$?

c) Find orthonormal bases of the four fundamental subspaces of A.

d) What is the singular value decomposition of A^T?

e) What is the pseudoinverse of A?

Solution.

a) A has rank 3.

b) $\|Av_1\| = 4$.

c) $\mathrm{Nul}(A)\colon \{v_4, v_5\}$ $\quad \mathrm{Col}(A)\colon \{u_1, u_2, u_3\}$ $\quad \mathrm{Nul}(A^T)\colon \{u_4\}$ $\quad \mathrm{Row}(A)\colon \{v_1, v_2, v_3\}$

d) $A^T = V\Sigma^T U^T$

e) $$A^+ = V \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U^T$$

Problem 3.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}.$$

a) Find a permutation matrix P, a lower-unitriangular matrix L, and an upper-triangular matrix U such that $PA = LU$.

b) Use **a)** to solve $Ax = b$, for $b = \begin{pmatrix} 12 \\ 1 \\ -30 \\ 6 \end{pmatrix}$.

c) What is $\det(A)$?

Solution.

a) $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -4 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

b) $x = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}$

c) $\det(A) = 1$

Problem 4. [20 points]

Consider the subspace

$$W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

a) Compute an orthonormal basis for W. [Hint: W is *not* all of $\mathbf{R}^3$.]

b) What is $\dim(W)$?

c) Compute the matrix P for orthogonal projection onto W. (You may write P as a product of two matrices, without expanding.)

d) Write an eigenvector of P.

e) Find the distance from $\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ to W.

f) Compute a basis for $W^\perp$.

Solution.

a)

$$\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

b)

$$\dim(W) = 2$$

c)

$$P = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{5} \\ -1/\sqrt{6} & 0 \\ -2/\sqrt{6} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{pmatrix}$$

d) Any nonzero vector in W; for instance, $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

e) Noting that

$$\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$$

shows $(1, 5, -2)$ is in Nul(A), so its projection is zero, and hence the distance is just

$$\left\| \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \right\| = \sqrt{30}.$$

f) We showed above that the vector $(1, 5, -2)$ is in $W^\perp$. Since $\dim(W) = 2$ we have $\dim(W^\perp) = 1$, so it is a basis.

Problem 5.

[20 points]

Consider the data points

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}.$$

a) Form the matrix A_0 with the data points as columns, and form the matrix A by subtracting the row averages from A_0.

b) Find the eigenvalues and eigenvectors of $S = \frac{1}{3}AA^T$.

c) Find the line closest to the columns of A.

d) Find the plane closest to the columns of A.

e) Find the plane closest to the original data points.

Solution.

a)

$$A_0 = \begin{pmatrix} 3 & 1 & 2 & 2 \\ 2 & 2 & 4 & 0 \\ -2 & -2 & -4 & -4 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

b) The matrix

$$S = \frac{1}{3}AA^T = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

is diagonal. It has eigenvalues $8/3, 4/3, 2/3$ with eigenvectors e_2, e_3, e_1, respectively.

c) The closest line is spanned by e_2: it is the y-axis.

d) The closest plane is spanned by e_2, e_3: it is the yz-plane.

e) We need to add back the row averages:

$$\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \operatorname{Span}\{e_2, e_3\}.$$

Problem 6. [30 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A 4×4 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$.

b) A 4×6 matrix of rank 6.

c) A 2×2 matrix whose column space is the line $3x + y = 0$ and with null space $\{0\}$.

d) A 2×2 matrix A that is not diagonalizable over $\mathbf{C}$, such that A^2 is diagonalizable.

e) A 3×4 matrix with singular values 2 and 1.

f) A positive-semidefinite symmetric matrix that is not positive-definite.

g) A matrix of rank 1 that cannot be written as a product of a column vector and a row vector.

h) A nonzero symmetric matrix with characteristic polynomial $p(\lambda) = \lambda^2$.

i) A matrix A satisfying

$$\dim\big(\mathrm{Row}(A)^\perp\big) = 2 \quad \text{and} \quad \dim\big(\mathrm{Col}(A)^\perp\big) = 3.$$

j) A 3×3 matrix with no real eigenvalues.

Solution.

a) One example is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

b) Does not exist: the rank is at most 4.

c) Does not exist: $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = 2$.

d) One example is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

e) One example is $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

f) One example is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

g) Does not exist by the outer product form of the SVD.

h) Does not exist: any such matrix equals QDQ^T for $D = 0$.

i) One example is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

j) Does not exist: every cubic polynomial has a real root.

Problem 7. [10 points]

Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "the columns of A are linearly independent?" Circle all that apply.

(1) A has full column rank.
(2) $Ax = b$ has a unique solution for every b in $\mathbf{R}^m$.
(3) $Ax = b$ has a unique least-squares solution for every b in $\mathbf{R}^m$.
(4) $Ax = 0$ has a unique solution.
(5) A has n pivots.
(6) $\mathrm{Nul}(A) = \{0\}$.
(7) $m \geq n$.
(8) $A^T A$ is invertible.
(9) AA^T is invertible.
(10) $A^+ A$ is the identity matrix.
(11) $\mathrm{Row}(A) = \mathbf{R}^n$.

Solution.

(1), (3), (4), (5), (6), (8), (10), (11)

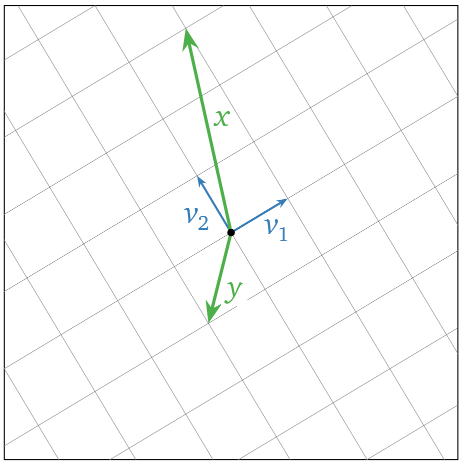
Problem 8.

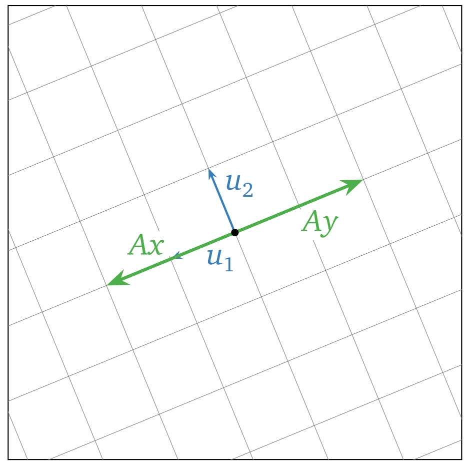
[10 points]

A certain 2×2 matrix A has the singular value decomposition

$$A = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^T,$$

where u_1, u_2, v_1, v_2 are drawn in the diagrams below. Given x and y in the diagram on the left, draw Ax and Ay on the diagram on the right.





Problem 9.

[15 points]

A certain diagonalizable 2×2 matrix A is equal to CDC^{-1}, where C has columns w_1, w_2 pictured below, and $D = \begin{pmatrix} 2 & 0 \\ 0 & 1/4 \end{pmatrix}$.

a) Draw $C^{-1}v$ on the left.

b) Draw $DC^{-1}v$ on the left.

c) Draw $Av = CDC^{-1}v$ on the right.