MATH 218D
PRACTICE MIDTERM EXAMINATION 1

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. All other materials are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- For full credit you must show your work so that your reasoning is clear.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ______________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ______________________________ Time: ______________

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.

Problem 1. [25 points]

Consider

$$A = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 12 \\ 1 \\ -30 \\ 6 \end{pmatrix}.$$

The goal of this question is to solve the equation $Ax = b$ like a computer would.

a) Carry out Gaussian reduction with maximal partial pivoting to find a $PA = LU$ decomposition. You should obtain

$$U = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 4 & -9 \\ 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix}.$$

Be sure to specify what L and P are.

b) Solve the equations $Ly = Pb$ and $Ux = y$ to find a solution of $Ax = b$.

Solution.

a) $$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & -\frac{1}{4} & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

b) $$Ly = Pb \implies y = \begin{pmatrix} 12 \\ -30 \\ 48 \\ 1 \end{pmatrix} \qquad Ux = y \implies x = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}.$$

Problem 2. [5 points]

Consider the matrix

$$B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 7 & 1 \\ 2 & 4 & 1 \end{pmatrix}.$$

Is B invertible? If so, find its inverse. If not, explain why.

Solution.

The matrix is not invertible: a row echelon form is $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which has only two pivots.

Problem 3. [20 points]

Consider the matrix

$$D = \begin{pmatrix} 1 & 2 & 3 & 2 & 14 & 9 \\ 0 & 0 & 0 & 2 & 10 & 6 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Note that this matrix is in row echelon form.

a) Fill in the blanks:

(1) The column space Col(D) is a subspace of $\mathbf{R}^m$, where $m = \boxed{}$.

(2) The null space Nul(D) is a subspace of $\mathbf{R}^n$, where $n = \boxed{}$.

b) Write down a vector b such that $Dx = b$ has no solution. If no such vector exists, explain why not.

c) Compute the reduced row echelon form of D.

d) Find a set of vectors that spans Nul(D).

e) Compute the solutions of $Dx = (2,2,0)$, noting that $D(0,0,0,1,0,0) = (2,2,0)$.

Solution.

a) Col(D) is a subspace of $\mathbf{R}^3$ and Nul(D) is a subspace of $\mathbf{R}^6$.

b) No such vector exists: D has full row rank.

c) $\begin{pmatrix} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

d) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$

e) $\text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Problem 4. [5 points]

Decide if a matrix with the following properties has full row rank, full column rank, both, or neither.

a) $Ax = b$ has 0 or 1 solutions, depending on b.

b) $Ax = b$ has 1 solution for every b.

c) $Ax = b$ has 0 or ∞ solutions, depending on b.

d) $Ax = b$ has ∞ solutions, for every b.

Solution.

a) Full column rank, not full row rank.

b) Both.

c) Neither.

d) Full row rank, not full column rank.

Problem 5. [10 points]

Consider the subspace V and vectors b_1 and b_2:

$$V = \text{Span}\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}2\\-1\\-1\end{pmatrix}, \begin{pmatrix}3\\-5\\-4\end{pmatrix}\right\} \qquad b_1 = \begin{pmatrix}-2\\8\\6\end{pmatrix} \qquad b_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

a) Is b_1 contained in V? If so, write b_1 as a linear combination of the vectors in the span; if not, explain why.

b) Same question for b_2.

c) Circle one: V is a

point line plane space

Why?

Solution.

a) Yes. There are many ways to write b_1 as a linear combination of the vectors; one is

$$b_1 = 2\begin{pmatrix}1\\3\\2\end{pmatrix} - 2\begin{pmatrix}2\\-1\\-1\end{pmatrix} + 0\begin{pmatrix}3\\-5\\-4\end{pmatrix}.$$

b) No, the system is inconsistent:

$$\left(\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 3 & -1 & -5 & 1\\ 2 & -1 & -4 & 1\end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & -7 & -14 & -2\\ 0 & 0 & 0 & \frac{3}{7}\end{array}\right).$$

c) V is a plane: it contains two noncollinear vectors, but it is not all of $\mathbf{R}^3$.

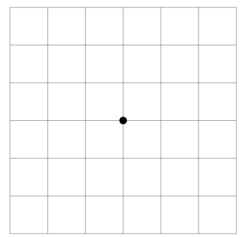
Problem 6.

[10 points]

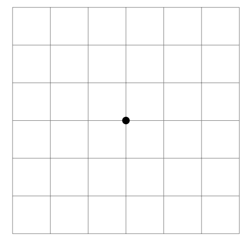
Give examples of 2×2 matrices A, B, C with ranks $0, 1$, and 2, respectively, and draw pictures of the null space and column space. (Be precise!)

a) Rank 0: $A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

Nul(A)

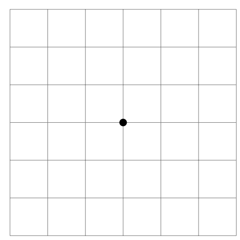


Col(A)

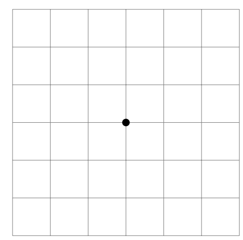


b) Rank 1: $B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

Nul(B)

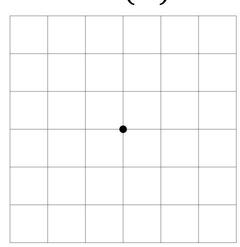


Col(B)

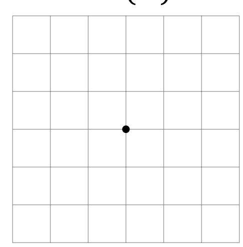


c) Rank 2: $C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

Nul(C)



Col(C)

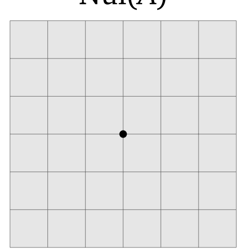


Solution.

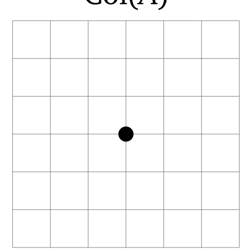
There are many possible answers for **b)** and **c)**.

a) Rank 0: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Nul(A)

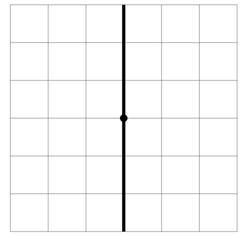


Col(A)

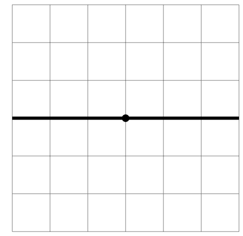


b) Rank 1: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Nul(A)

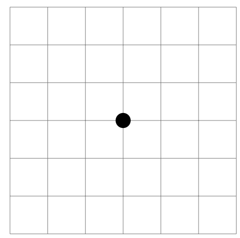


Col(A)



c) Rank 2: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Nul(A)



Col(A)