MATH 218D
MIDTERM EXAMINATION 1

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

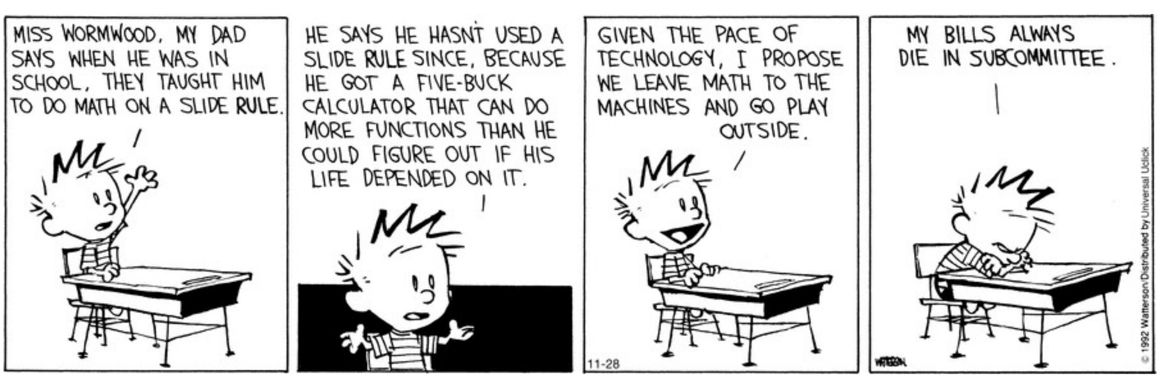
- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear.
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ______________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ______________________________ Time: ______________



Problem 1. [20 points]

Consider

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -2 & -1 & -1 & 2 \\ 2 & 3 & 5 & 4 \\ 6 & 3 & -3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ -8 \\ 4 \\ 0 \end{pmatrix}.$$

a) Carry out Gaussian reduction with maximal partial pivoting to find a $PA = LU$ decomposition. You should obtain

$$U = \begin{pmatrix} 6 & 3 & -3 & 0 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Be sure to specify what L and P are. Please write the row operations you performed.

b) Solve the equations $Ly = Pb$ and $Ux = y$ to find a solution of $Ax = b$.

c) Briefly explain why step **b)** is faster than solving $Ax = b$ using Gaussian elimination on the augmented matrix $\left(A \mid b\right)$, once you have a $PA = LU$ decomposition.

Solution.

a)

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

b)

$$Ly = Pb \implies y = \begin{pmatrix} 0 \\ 4 \\ -4 \\ -6 \end{pmatrix} \qquad Ux = y \implies x = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$

c) Gaussian elimination takes about $\frac{2}{3} \cdot 4^3 \approx 43$ flops, whereas forward- and back-substitution take about $4^2 = 16$ flops.

Problem 2. [15 points]

a) Compute the inverse of $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}$.

Be sure to write out any row operations you perform.

b) For which value(s) of k is $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & k \\ 2 & 3 & 9 \end{pmatrix}$ *not* invertible?

c) Suppose that A is a 3×3 matrix whose third column is in the span of the first two. Briefly explain why A is not invertible.

[**Hint:** can it have full row rank?]

Solution.

a)

$$\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -69 & -27 & 8 \\ -8 & -3 & 1 \\ 18 & 7 & -2 \end{pmatrix}$$

b)

$$k = -\frac{36}{7}$$

c) The column space of A is a plane in $\mathbf{R}^2$ (or a line, or a point), so it does not have full row rank, and hence has fewer than 3 pivots.

Problem 3. [25 points]

Consider

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 \\ -2 & -6 & 6 & -2 \\ 2 & 6 & 3 & -7 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ -8 \\ -10 \end{pmatrix}.$$

a) Find the parametric vector form of the solution set of $Ax = b$. Be sure to write out any row operations you perform.

b) Write down two different solutions of $Ax = b$. (Your answer will be two vectors with numbers in them.)

c) Does $Ax = b'$ have a solution for every vector $b' \in \mathbf{R}^3$? Why or why not?

d) Find a spanning set for $\mathrm{Nul}(A)$.

e) Let $v = (-1, 1, 1, 1)$. Check that $v \in \mathrm{Nul}(A)$, and write v as a linear combination of the spanning vectors you obtained in **d)**.

[**Hint:** what values do the free variables have to take?]

Solution.

a)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

b) Choose any values of the free variables. For instance, $(x_2, x_4) = (1, 0)$ and $(0, 1)$ give $(-5, 1, -2, 0)$ and $(0, 0, -1, 1)$, respectively.

c) No: the matrix A has only two pivots, hence does not have full row rank.

d)

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

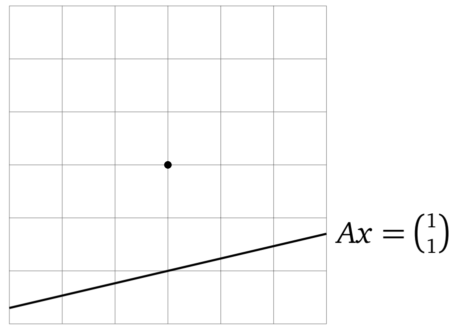
e) One computes $Av = 0$, so $v \in \mathrm{Nul}(A)$. The second (resp. fourth) coordinate of v is the value of x_2 (resp. x_4), so

$$v = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 4.

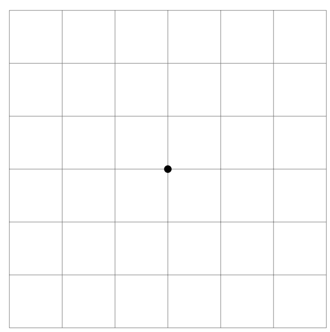
[20 points]

For a certain 2×2 matrix A, the solution set of $Ax = \binom{1}{1}$ is drawn. Copy this grid onto your paper, and draw **a)** the solution set of $Ax = 0$ and **b)** the solution set of $Ax = \binom{-1}{-1}$. Be sure to label which is which.



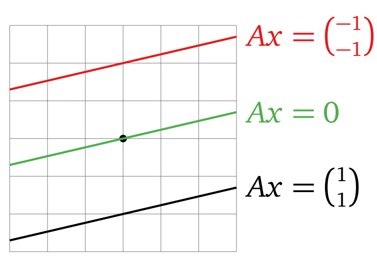
c) What is the rank of A?

d) Draw the column space of A in a grid like below. Be precise!



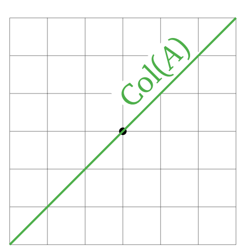
Solution.

a) and **b)**



c) The rank is 1, since A has one free variable.

d)



Problem 5. [15 points]

Find examples of matrices with the following properties. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead.

a) A matrix A, in RREF, such that $Ax = b$ has at least one solution for every b, but A does not have full column rank.

b) A 3×5 matrix of rank 4, in RREF.

c) A 2×2 matrix A such that the solution set of $Ax = \binom{3}{4}$ is a line, and $Ax = \binom{1}{1}$ has no solutions.

d) A 3×3 matrix A with no zero entries, such that $\mathrm{Col}(A)$ is a plane.

e) A 4×4 matrix A with full row rank such that $A(1, 2, -1, 1) = 0$.

Solution.

a) There are many answers; one is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

b) Yeah, right.

c) There are many answers; one is $A = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$.

d) There are many answers; one is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

e) As if.

Problem 6. [18 points]

Which of the following are subspaces of $\mathbf{R}^4$? If not, why?

a) $\text{Span}\left\{\begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}0\\-2\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\-1\\2\end{pmatrix}\right\}$

b) $\text{Nul}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

c) $\text{Col}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

d) $\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\}$

e) $\{\}$

f) $V = \left\{\text{all vectors } \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \text{ such that } xy = zw\right\}$

Solution.

a) Yes.

b) No: this is a subspace of $\mathbf{R}^3$.

c) Yes.

d) Yes.

e) No: this does not contain the zero vector.

f) No: this is not closed under addition. For instance,

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}.$$