MATH 218D
PRACTICE MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear.
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ______________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ______________________________ Time: ______________

> This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.

Problem 1.

[25 points]

For a certain 3×4 matrix A, the reduced row echelon form of $(A \mid I_3)$ is

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & -3 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

a) Compute a basis for $\mathrm{Nul}(A)$.

b) Compute a basis for $\mathrm{Row}(A)$.

c) Compute a basis for $\mathrm{Nul}(A^T)$.

d) Compute a basis for $\mathrm{Col}(A)$.
[**Hint:** $\mathrm{Col}(A) = \mathrm{Nul}(A^T)^\perp$.]

e) What is the rank of A? What are the dimensions of its four fundamental subspaces?

f) Compute the projection matrix onto $\mathrm{Nul}(A^T)$.

Solution.

$$\textbf{a)}\ \left\{\begin{pmatrix}-2\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\3\\1\end{pmatrix}\right\} \qquad \textbf{b)}\ \left\{\begin{pmatrix}1\\2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-3\end{pmatrix}\right\} \qquad \textbf{c)}\ \left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}\right\}$$

d) The orthogonal complement of $\mathrm{Nul}(A^T)$ is the null space of $\begin{pmatrix}1 & -1 & 2\end{pmatrix}$, which has basis

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}.$$

e) The rank of A is 2, and

$$\dim \mathrm{Nul}(A) = 2 \quad \dim \mathrm{Row}(A) = 2 \quad \dim \mathrm{Nul}(A^T) = 1 \quad \dim \mathrm{Col}(A) = 2.$$

f) We use the formula for projection onto the line spanned by $u = (1, -1, 2)$:

$$P_{\mathrm{Nul}(A)} = \frac{1}{u \cdot u} u u^T = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}.$$

Problem 2. [25 points]

a) Give an example of a 3×2 matrix A and a vector $b \in \mathbf{R}^3$ such that the equation $Ax = b$ is inconsistent but has infinitely many least-squares solutions.

b) Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^2.$$

c) If A and B are $n \times n$ matrices such that $ABx = 0$ has only one solution, does $Ax = 0$ necessarily have only one solution? Why or why not?

d) Let P_V be the matrix for projection onto a plane V in $\mathbf{R}^3$. What is the rank of P_V? What is its determinant?

e) Explain why any set containing the zero vector is linearly dependent.

Solution.

a) There are many examples; one is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

b)

$$\det \begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^2 = (-10)^2 = 100$$

c) If $ABx = 0$ has only one solution then AB is invertible, so $0 \neq \det(AB) = \det(A)\det(B)$, so $\det(A) \neq 0$, so A is invertible, and hence $Ax = 0$ has only one solution.

d) The rank is 2 and the determinant is 0 (P_V is not invertible).

e) Because $1 \cdot 0 + 0 \cdot v_1 + \cdots + 0 \cdot v_n$ is a linear dependence relation.

Problem 3. [30 points]

Consider the matrix

$$A = \begin{pmatrix} 3 & -3 \\ 3 & 1 \\ 3 & 5 \\ 3 & 1 \end{pmatrix}.$$

Let $V = \mathrm{Col}(A)$.

a) Compute the QR decomposition of A.

b) What is an orthonormal basis $\{u_1, u_2\}$ for V?

c) Compute the projection matrix P_V onto V.

d) Compute the orthonogal decomposition of $b = (1, 0, 1, 0)$ with respect to V.

e) Compute an orthonormal basis $\{w_1, w_2\}$ of $V^\perp$.

f) Explain why $\{u_1, u_2, w_1, w_2\}$ is an orthonormal basis of $\mathbf{R}^4$.

Solution.

a) $Q = \begin{pmatrix} 1/2 & -1/\sqrt{2} \\ 1/2 & 0 \\ 1/2 & 1/\sqrt{2} \\ 1/2 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 6 & 2 \\ 0 & 4\sqrt{2} \end{pmatrix}$ **b)** $\left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$

c) $P_V = QQ^T = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$

e) The orthogonal complement of V is $\mathrm{Nul}\begin{pmatrix} 3 & 3 & 3 & 3 \\ -3 & 1 & 5 & 1 \end{pmatrix}$. We compute a basis and run Gram–Schmidt:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \rightsquigarrow \left\{ \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \right\}.$$

f) The set $\{u_1, u_2, w_1, w_2\}$ is orthonormal since each $u_i \cdot w_j = 0$. Hence it is linearly independent, so it is a basis for $\mathbf{R}^4$.

Problem 4. [15 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 2 \\ 8 \\ 4 \end{pmatrix}$$

a) Find a linear dependence relation among $\{v_1, v_2, v_3, v_4\}$.

b) What is the dimension of $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$?

c) Explain why any set of four vectors in $\mathbf{R}^3$ is linearly dependent.

Solution.

a) $14v_1 - 4v_2 - 4v_3 + v_4 = 0$

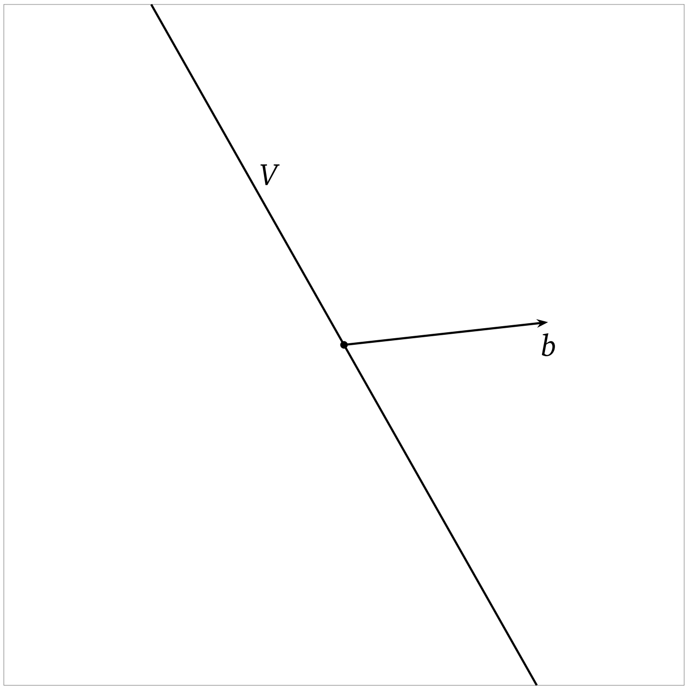
b) The dimension is 3.

c) A wide matrix has a free column.

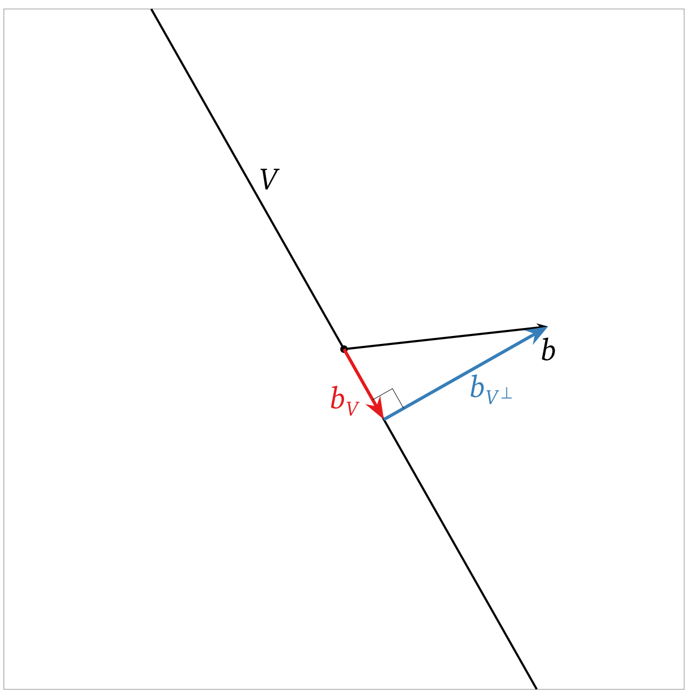
Problem 5.

[10 points]

A subspace V and a vector b are drawn below. Draw the projection b_V of b onto V, and draw the projection $b_{V^\perp}$ of b onto $V^\perp$. Label your answers!



Solution.

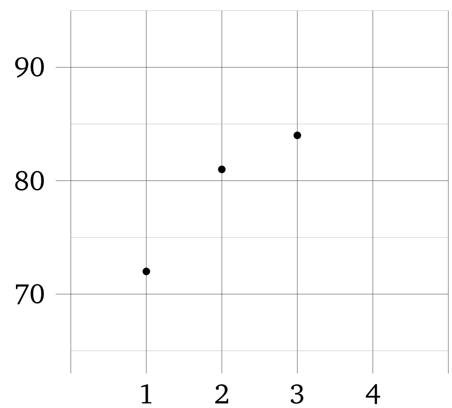


Problem 6. [15 points]

Your blockmate Karxon is currently taking Math 105L. Karxon scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Karxon does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) The general equation of a line in $\mathbf{R}^2$ is $y = Cx + D$. Write down the system of linear equations in C and D that would be satisfied by a line passing through the points $(1, 72)$, $(2, 81)$, and $(3, 84)$, and then write down the corresponding matrix equation.

b) Solve the corresponding least squares problem for C and D, and use this to *write down* and *draw* the the best fit line below. [Use a calculator]



$$y = \square\, x + \square$$

c) What score does this line predict for the fourth (final) exam?

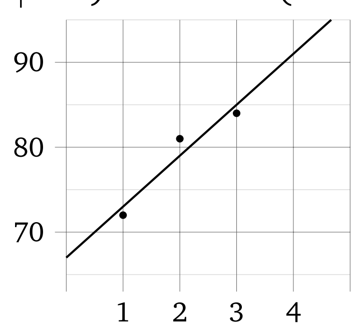
Solution.

a)

$$\begin{aligned} 1C + D &= 72 \\ 2C + D &= 81 \\ 3C + D &= 84 \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix}$$

b) The normal equation is $A^T A x = A^T b$, which gives rise to the following augmented matrix:

$$\left(\begin{array}{cc|c} 14 & 6 & 486 \\ 6 & 3 & 237 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 67 \end{array}\right)$$



$$y = 67 + 6x$$

c) $67 + 6 \cdot 4 = 91$