MATH 218D
MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear. (You do not need to show your steps in Gauss–Jordan elimination.)
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ____________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ____________________________ Time: ______________

Problem 1.

[20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) $\text{Nul}\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix}$

b) $\text{Col}\begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \\ 4 & -1 & 11 \end{pmatrix}$

c) The subspace of all vectors in $\mathbf{R}^4$ whose entries sum to zero.

d) The line $\{(t, 2t, 3t) : t \in \mathbf{R}\}$.

e) $\mathbf{R}^3$

Solution.

These are the bases you would obtain if you did the problem the same way I did.

a) $\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix}\right\}$ **b)** $\left\{\begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ 0 \\ 1 \end{pmatrix}\right\}$ **c)** $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

d) $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$ **e)** $\{\}$

Problem 2.

[25 points]

In this problem we will consider the best-fit plane $z = Bx + Cy + D$ through the data points

$$\begin{pmatrix} 3 \\ -5 \\ b_1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ b_2 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 5 \\ b_3 \end{pmatrix} \quad \begin{pmatrix} 3 \\ -7 \\ b_4 \end{pmatrix}.$$

a) Compute the matrix A such that the coefficient vector $\hat{x} = (B, C, D)$ is the least-squares solution of

$$A \begin{pmatrix} B \\ C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

b) Compute the QR decomposition of A.

c) Given a vector $b = (b_1, b_2, b_3, b_4)$, explain how a computer would quickly compute $\hat{x}$ using the QR decomposition you found in **b)**.

d) Compute the best-fit plane $z = Bx + Cy + D$ when $(b_1, b_2, b_3, b_4) = (10, -10, 20, -20)$, using your QR decomposition or otherwise.

e) What is the minimum value of

$$\big(z(3,-5) - 10\big)^2 + \big(z(1,1) + 10\big)^2 + \big(z(-1,5) - 20\big)^2 + \big(z(3,-7) + 20\big)^2$$

for all planes $z = Bx + Cy + D$?

Solution.

a)
$$A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & 1 \\ 3 & -7 & 1 \end{pmatrix}$$

b)
$$Q = \begin{pmatrix} 3/2\sqrt{5} & 1/2\sqrt{5} & -1/\sqrt{10} \\ 1/2\sqrt{5} & 3/2\sqrt{5} & -1/\sqrt{10} \\ -1/2\sqrt{5} & 3/2\sqrt{5} & 2/\sqrt{10} \\ 3/2\sqrt{5} & -1/2\sqrt{5} & 2/\sqrt{10} \end{pmatrix} \qquad R = \begin{pmatrix} 2\sqrt{5} & -4\sqrt{5} & 3\sqrt{5}/5 \\ 0 & 2\sqrt{5} & 3\sqrt{5}/5 \\ 0 & 0 & \sqrt{10}/5 \end{pmatrix}$$

c) A computer would solve $R\hat{x} = Q^T b$ by back-substitution.

d) $z = 3x + 3y$

e) This is $\|b - \hat{b}\|^2 = 640$.

Problem 3. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & k \\ -2 & 0 & 1 & 1 \\ k & 2 & 1 & 3 \\ 2 & 1 & -1 & 1 \end{pmatrix}.$$

a) Compute the determinant of A. For what value(s) of k (a real number), if any, is A *not* invertible?

b) Compute $\det(A^T A)$.

c) Compute $\det(BAB^{-1})$ where B is a 4×4 invertible matrix.

Solution.

a) $\det(A) = -(k^2 + 6)$. This is nonzero for all real numbers k.

b) $\det(A^T A) = (k^2 + 6)^2$.

c) $\det(BAB^{-1}) = \det(A) = -(k^2 + 6)$.

Problem 4. [20 points]

Consider the subspace V in $\mathbf{R}^4$ defined by $x_1 + 2x_2 - 2x_3 - x_4 = 0$.

a) Compute the orthogonal projection $b_{V^\perp}$ of the vector $b = (0, -3, 3, -2)$ onto $V^\perp$.

b) Compute the orthogonal decomposition $b = b_V + b_{V^\perp}$.

c) Find the matrix P_V for orthogonal projection onto V.

d) Find a basis for $\mathrm{Nul}(P_V)$.

Solution.

a) $b_{V^\perp} = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix}$ **b)** $b = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix}$

c) $P = \dfrac{1}{10}\begin{pmatrix} 9 & -2 & 2 & 1 \\ -2 & 6 & 4 & 2 \\ 2 & 4 & 6 & -2 \\ 1 & 2 & -2 & 9 \end{pmatrix}$ **d)** $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \right\}$

Problem 5. [16 points]

a) Let A be an $m \times n$ matrix of rank r. Which of the following statements are equivalent to "A has full row rank"?
(1) $\mathrm{Nul}(A^T) = \{0\}$
(2) $n = r$
(3) $\mathrm{Col}(A) = \mathbf{R}^m$
(4) A has linearly independent columns
(5) A has a pivot in every row
(6) A is invertible
(7) $Ax = b$ is consistent for every vector b

b) Explain why the projection matrix P_V onto a subspace V can be written as QQ^T for some matrix Q with orthonormal columns.

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly *de*pendent, but v_3 is not in $\mathrm{Span}\{v_1, v_2\}$. Be sure to label which is v_3.

d) Give an example of a 4×4 matrix A such that $\mathrm{Nul}(A) = \mathrm{Row}(A)$, or explain why no such matrix exists.

Solution.

a) (1),(3),(5),(7)

b) Choose an orthonormal basis for V, then let Q be the matrix with those columns.

c) There are many answers. One is

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

d) This is impossible since $\mathrm{Nul}(A) = \mathrm{Row}(A)^\perp$.

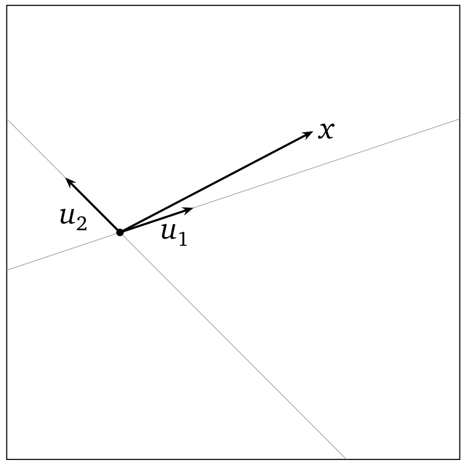
Problem 6. [10 points]

Unit vectors u_1 and u_2 and a vector x are drawn in the picture below. Copy this picture onto your page as best you can, then:

a) Draw and label $(x \cdot u_1)u_1$ and $(x \cdot u_2)u_2$.

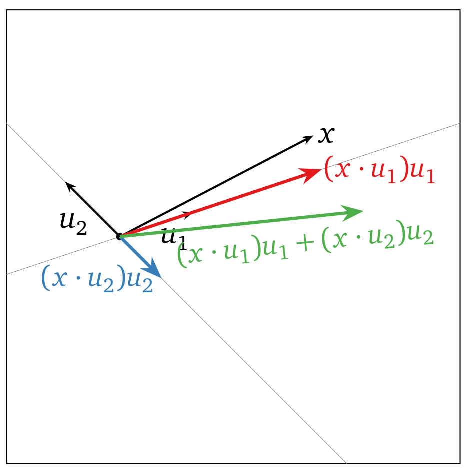
b) Draw and label $(x \cdot u_1)u_1 + (x \cdot u_2)u_2$.

c) Note that $\{u_1, u_2\}$ is a basis for $V = \mathbf{R}^2$. Explain why $x = x_V \neq (x \cdot u_1)u_1 + (x \cdot u_2)u_2$ does not contradict the projection formula.



Solution.

a),b)



c) The vectors u_1, u_2 are not orthogonal.