MATH 218D
PRACTICE MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear.
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ______________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ______________________________ Time: ______________

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.

Problem 1. [20 points]

Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}^T.$$

From this you can read off all of the following properties of A *without* computing A.

a) A is a $\boxed{} \times \boxed{}$ matrix of rank $r = \boxed{}$.

b) Find orthonormal bases of the four fundamental subspaces of A.

c) Compute the matrix P_V for orthogonal projection onto $V = \mathrm{Col}(A)$ (write it as a product, without expanding it out).

d) Write the SVD of A in vector form.

e) Find an orthogonal diagonalization $A^T A = QDQ^T$.

Solution.

a) The size and rank of A can be read off from Σ: A is a 4×3 matrix of rank 2.

b) The columns of U and V give orthonormal bases for the four subspaces.

$$\mathrm{Nul}(A)\colon \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \mathrm{Row}(A)\colon \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},\ \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

$$\mathrm{Col}(A)\colon \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix},\ \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \right\} \qquad \mathrm{Nul}(A^T)\colon \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

c) We have an orthonormal basis for $\mathrm{Col}(A)$ from **b)**; putting these vectors in a matrix Q, we have $P_V = QQ^T$ for

$$Q = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} \\ -2/\sqrt{10} & 3/\sqrt{15} \\ 2/\sqrt{10} & 2/\sqrt{15} \\ -1/\sqrt{10} & -1/\sqrt{15} \end{pmatrix}$$

d) This is the vector form of the SVD:

$$A = 3 \cdot \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} + 2 \cdot \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & -2 & -1 \end{pmatrix}$$

e) We can take

$$Q = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix} \qquad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 2. [20 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

a) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

b) Which of these adjectives describe S?
- positive-definite
- positive-semidefinite
- negative-definite
- negative-semidefinite
- indefinite

c) Write the singular value decomposition of S in matrix form.

d) Find the maximum value of the quadratic form $q(x) = x^T S x$ subject to $\|x\| = 1$. At which vectors is this value obtained?

Solution.

a) First we compute the eigenvalues of S. We find the characteristic polynomial by expanding cofactors along the first column:

$$\begin{aligned} p(\lambda) = \det\begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{pmatrix} \\ &= (1-\lambda)\det\begin{pmatrix} -1-\lambda & -2 \\ -2 & -\lambda \end{pmatrix} + 2\det\begin{pmatrix} 0 & 2 \\ -1-\lambda & -2 \end{pmatrix} \\ &= (1-\lambda)\big[(-1-\lambda)(-\lambda) - 4\big] + 2\big[-2(-1-\lambda)\big] \\ &= (1-\lambda)(\lambda^2+\lambda-4) - 4(-1-\lambda) \\ &= \lambda^2 + \lambda - 4 - \lambda^3 - \lambda^2 + 4\lambda + 4 + 4\lambda \\ &= -\lambda^3 + 9\lambda = -\lambda(\lambda-3)(\lambda+3). \end{aligned}$$

The eigenvalues are 0 and ± 3; we compute eigenvectors:

$$\lambda = 3: \quad S - 3I = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 2 \\ 2 & -2 & -3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow v_1 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\lambda = -3: \quad S + 3I = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & -2 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow v_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda = 0: \quad S - 0I = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow v_3 = \frac{1}{3}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

Hence $S = QDQ^T$ for

$$Q = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) S has a positive and a negative eigenvalue, so it is indefinite.

c) The singular values of S are the absolute values of the nonzero eigenvalues: $\sigma_1 = \sigma_2 = 3$. We have

$$\begin{aligned} 3v_1 = Sv_1 = \sigma_1 u_1 &\implies u_1 = v_1 \\ -3v_2 = Sv_2 = \sigma_2 u_2 &\implies u_2 = -v_2. \end{aligned}$$

We can take $u_3 = v_3$ as our orthonormal basis of $\mathrm{Nul}(S) = \mathrm{Nul}(S^T)$, so

$$S = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -1 & -2 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{pmatrix}^T.$$

d) The maximum value is $\lambda_1 = 3$. It is achieved at $\pm v_1 = \pm\frac{1}{3}(2, -1, 2)$.

Problem 3.

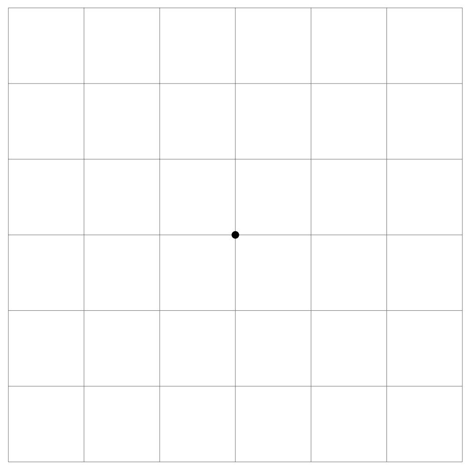
[20 points]

Consider the difference equation $v_{n+1} = Av_n$ for

$$A = \begin{pmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

a) Find a closed formula for $A^n v_0$ for $v_0 = \binom{1}{2}$. What happens when $n \to \infty$?

b) In the diagram, *draw and label* the eigenspaces of A, and draw the vectors $v_0, v_1, v_2, v_3, \ldots$ as points. (The grid lines are one unit apart, and the dot is the origin.)
[**Hint:** you do not have to compute $A^n v_0$ numerically to do this.]



c) Solve the system of ordinary differential equations

$$\begin{aligned} u_1' &= 2u_1 - u_2 & u_1(0) &= 1 \\ u_2' &= \tfrac{3}{2}u_1 - \tfrac{1}{2}u_2 & u_2(0) &= 2. \end{aligned}$$

Solution.

a) The characteristic polynomial of A is

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$. We compute eigenvectors:

$$A - 1I = \begin{pmatrix} 1 & -1 \\ \text{---} & \text{---} \end{pmatrix} \rightsquigarrow w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A - \frac{1}{2}I = \begin{pmatrix} \frac{3}{2} & -1 \\ \text{---} & \text{---} \end{pmatrix} \rightsquigarrow w_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence $A = CDC^{-1}$ for

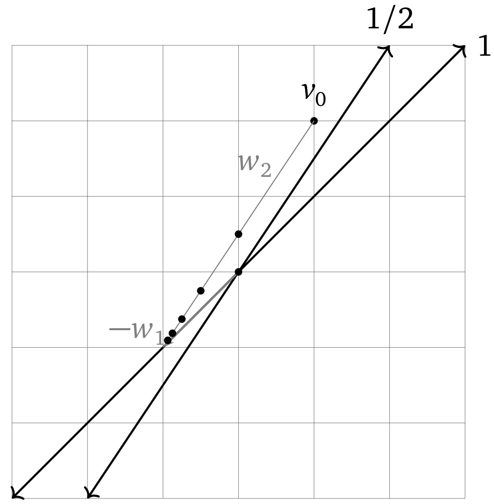
$$C = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

We eyeball $v_0 = -w_1 + w_2$, so

$$A^n v_0 = -w_1 + \frac{1}{2^n} w_2.$$

This approaches $-w_1 = \binom{-1}{-1}$ as $n \to \infty$.

b) The eigenspaces are spanned by w_1 and w_2.



c) We need to solve $u' = Au$ for $u(0) = \binom{1}{2} = v_0$ and A as above. We have $u(0) = -w_1 + w_2$, so the solution is

$$u(t) = -e^t w_1 + e^{t/2} w_2 \qquad \Longrightarrow \qquad \begin{aligned} u_1 &= -e^t + 2e^{t/2} \\ u_2 &= -e^t + 3e^{t/2} \end{aligned}$$

Problem 4. [20 points]

All of the following statements are false. Provide a counterexample to each. You need not justify your answers.

a) The singular values of a diagonalizable, invertible 2×2 matrix are the absolute values of the eigenvalues.

b) If S is symmetric, then either S or $-S$ is positive-semidefinite.

c) If $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ and $x \neq 0$, then $\|A^n x\| \to \infty$ as $n \to \infty$.

d) If λ is an eigenvalue of AA^T, then λ is an eigenvalue of A^TA.

e) Any invertible matrix is diagonalizable.

Solution.

a) The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 1 and 2, so it is invertible and diagonalizable. However,

$$A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 4$; we have $p(1^2) = -1$ and $p(2^2) = -4$, so neither 1 nor 2 is a singular value of A.

b) Any symmetric matrix S with both positive and negative eigenvalues works:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

c) The characteristic polynomial if A is $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$. An eigenvector with eigenvalue 1 is $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; for this vector, we have $A^n x = x$, so the length does not grow.

d) This can only be false for $\lambda = 0$. Zero is an eigenvalue of A^TA (resp. AA^T) if and only if A has linearly dependent columns (resp. rows), so we need to find a matrix with linearly independent columns and linearly dependent rows. For instance:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

e) The shear matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

Problem 5. [16 points]

All of the following statements are true. Explain why in a sentence or two.

a) If A is a 3×3 matrix that has eigenvalues 1 and -1, both of algebraic multiplicity one, then A is diagonalizable (over the real numbers).

b) Let V be a subspace of $\mathbf{R}^n$ and let P_V be the matrix for projection onto V. Then P_V is diagonalizable.

c) Any eigenvector of A with nonzero eigenvalue is contained in the column space of A.

d) A positive definite symmetric matrix has positive numbers on the main diagonal.

Solution.

a) The third eigenvalue is also a real number with multiplicity one.

b) The 1-eigenspace of P_V is V, and the 0-eigenspace is $V^\perp$. The geometric multiplicities add to n because $\dim(V) + \dim(V^\perp) = n$.

Alternatively, P_V is symmetric, so it is diagonalizable by the spectral theorem.

c) If $Ax = \lambda x$ for $\lambda \neq 0$ then $x = A\frac{1}{\lambda}x$ is a linear combination of the columns of A.

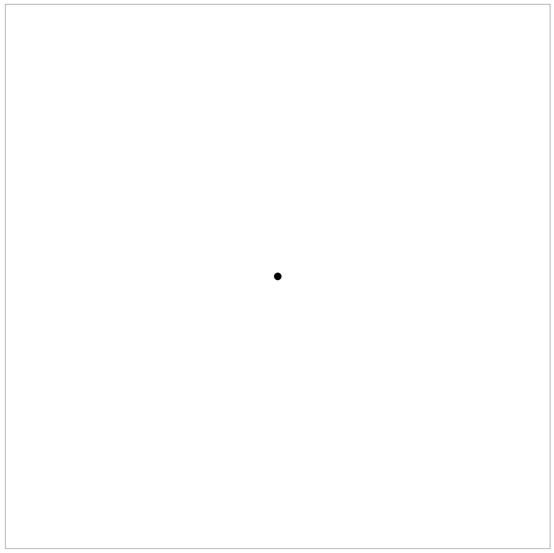
d) The (i,i) entry of a matrix S is $e_i^T S e_i$, which is positive if S is positive definite.

Problem 6. [10 points]

Draw a picture of the ellipse defined by the equation

$$30x_1^2 + 35x_2^2 + 12x_1x_2 = 1.$$

Be precise! Label your major and minor axes and radii.



Solution.

First we diagonalize the quadratic form $q(x_1, x_2) = 30x_1^2 + 35x_2^2 + 12x_1x_2$. We have $q(x) = x^T S x$ for $S = \begin{pmatrix} 30 & 6 \\ 6 & 35 \end{pmatrix}$; this matrix has orthogonal diagonalization $S = QDQ^T$ for $Q = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 39 & 0 \\ 0 & 26 \end{pmatrix}$. Hence our ellipse is obtained from the standard ellipse $39y_1^2 + 26y_2^2 = 1$ by multiplication by Q.