MATH 218D
PRACTICE MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear.
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ______________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ______________________________ Time: ______________

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.

Problem 1. [20 points]

Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}^T.$$

From this you can read off all of the following properties of A *without* computing A.

a) A is a ☐ $\times$ ☐ matrix of rank $r =$ ☐.

b) Find orthonormal bases of the four fundamental subspaces of A.

c) Compute the matrix P_V for orthogonal projection onto $V = \mathrm{Col}(A)$ (write it as a product, without expanding it out).

d) Write the SVD of A in vector form.

e) Find an orthogonal diagonalization $A^T A = QDQ^T$.

Problem 2.

[20 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

a) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

b) Which of these adjectives describe S?

- positive-definite
- positive-semidefinite
- negative-definite
- negative-semidefinite
- indefinite

c) Write the singular value decomposition of S in matrix form.

d) Find the maximum value of the quadratic form $q(x) = x^T S x$ subject to $\|x\| = 1$. At which vectors is this value obtained?

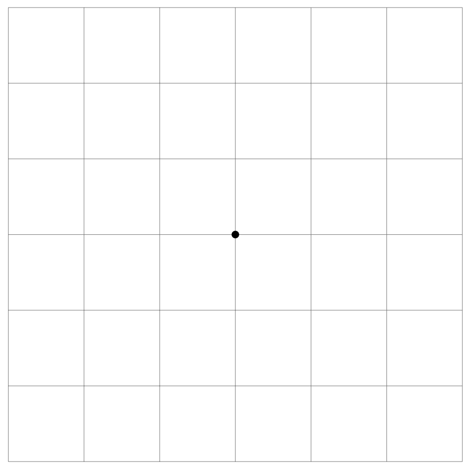
Problem 3. [20 points]

Consider the difference equation $v_{n+1} = Av_n$ for

$$A = \begin{pmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

a) Find a closed formula for $A^n v_0$ for $v_0 = \binom{1}{2}$. What happens when $n \to \infty$?

b) In the diagram, *draw and label* the eigenspaces of A, and draw the vectors $v_0, v_1, v_2, v_3, \ldots$ as points. (The grid lines are one unit apart, and the dot is the origin.) [**Hint:** you do not have to compute $A^n v_0$ numerically to do this.]



c) Solve the system of ordinary differential equations

$$\begin{aligned} u_1' &= 2u_1 - u_2 & u_1(0) &= 1 \\ u_2' &= \tfrac{3}{2}u_1 - \tfrac{1}{2}u_2 & u_2(0) &= 2. \end{aligned}$$

Problem 4. [20 points]

All of the following statements are false. Provide a counterexample to each. You need not justify your answers.

a) The singular values of a diagonalizable, invertible 2×2 matrix are the absolute values of the eigenvalues.

b) If S is symmetric, then either S or $-S$ is positive-semidefinite.

c) If $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ and $x \neq 0$, then $\|A^n x\| \to \infty$ as $n \to \infty$.

d) If λ is an eigenvalue of AA^T, then λ is an eigenvalue of $A^T A$.

e) Any invertible matrix is diagonalizable.

Problem 5. [16 points]

All of the following statements are true. Explain why in a sentence or two.

a) If A is a 3×3 matrix that has eigenvalues 1 and -1, both of algebraic multiplicity one, then A is diagonalizable (over the real numbers).

b) Let V be a subspace of $\mathbf{R}^n$ and let P_V be the matrix for projection onto V. Then P_V is diagonalizable.

c) Any eigenvector of A with nonzero eigenvalue is contained in the column space of A.

d) A positive definite symmetric matrix has positive numbers on the main diagonal.

Problem 6. [10 points]

Draw a picture of the ellipse defined by the equation

$$30x_1^2 + 35x_2^2 + 12x_1x_2 = 1.$$

Be precise! Label your major and minor axes and radii.