MATH 218D
MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- For full credit you must **show your work** so that your reasoning is clear. (You do not need to show your steps in Gauss–Jordan elimination.)
- If you need clarification or think you've found a typo, ask a **private question on Piazza**. We'll be monitoring it.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to to the Office of Student Conduct.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ________________________________ Time: ________________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ________________________________ Time: ________________

Problem 1. [20 points]

Consider the quadratic form

$$q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 8x_1x_3 + 8x_2x_3.$$

a) Find a symmetric matrix S such that $q(x) = x^T S x$.

b) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.
[**Hint:** One eigenvalue of S is 9.]

c) Find coordinates y_1, y_2, y_3 such that

$$q(x_1, x_2, x_3) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

(The y_i should be linear functions of the x_i.)

d) What are the minimum and maximum values of $q(x)$ subject to $\|x\| = 1$? For which values of x are those values attained?

Your answers should involve square roots and fractions, *not* decimals.

Solution.

a)

$$S = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & 4 \\ -4 & 4 & 5 \end{pmatrix}$$

b)

$$Q = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \qquad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

c)

$$y_1 = \frac{-x_1 + x_2 + 2x_3}{\sqrt{6}} \qquad y_2 = \frac{x_1 + x_2}{\sqrt{2}} \qquad y_3 = \frac{x_1 - x_2 + x_3}{\sqrt{3}}$$

d) The minimum value is -3, which is attained at $\pm\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The maximum value is 9, which is attained at $\pm\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Problem 2. [10 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

a) Verify that S is positive-definite without finding its eigenvalues.

b) Compute the LDL^T and Cholesky decompositions of S:

$$S = LDL^T \qquad S = L_1 L_1^T.$$

Solution.

a) This can be accomplished by finding the LU decomposition, which we do in **b)**.

b) We have $S = LDL^T$ for

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We also have $S = L_1 L_1^T$ for

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{3} \end{pmatrix}.$$

Problem 3. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

a) Compute the singular value decomposition of A in outer-product form:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T.$$

Clearly label each of $\sigma_1, \sigma_2, u_1, u_2, v_1$, and v_2.

b) Compute the singular value decomposition of A in matrix form:

$$A = U\Sigma V^T.$$

Clearly label U, Σ, and V.

c) Use your answer to **b)** to find orthonormal bases for the four subspaces of A.

Your answers should involve square roots and fractions, *not* decimals.

Solution.

a) We compute the SVD of A^T, as that is easier. We have $AA^T = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$ and corresponding unit eigenvectors $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Hence the singular values are $\sigma_1 = \sqrt{5}$ and $\sigma_2 = 1$, and we compute

$$v_1 = \frac{1}{\sqrt{5}} A^T u_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = A^T u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence the SVD is

$$A = \sqrt{5}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 2 & 2 & 1 \end{pmatrix} + 1\,\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}.$$

b) We also need an orthonormal basis of Nul(A). Finding the null space in parametric vector form and running Gram–Schmidt gives the orthonormal basis

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \xrightarrow{\text{Gram–Schmidt}} \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

for Nul(A). Hence $A = U\Sigma V^T$ for

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{5} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1/\sqrt{10} & -1/\sqrt{2} & 0 & 2/\sqrt{10} \\ 2/\sqrt{10} & 0 & -1/\sqrt{2} & -1/\sqrt{10} \\ 2/\sqrt{10} & 0 & 1/\sqrt{2} & -1/\sqrt{10} \\ 1/\sqrt{10} & 1/\sqrt{2} & 0 & 2/\sqrt{10} \end{pmatrix}.$$

c) We read off orthonormal bases for the four subspaces from **b)**:

$$\mathrm{Col}(A):\ \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\right\} \qquad \mathrm{Row}(A):\ \left\{\frac{1}{\sqrt{10}}\begin{pmatrix}1\\2\\2\\1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}$$

$$\mathrm{Nul}(A):\ \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\\0\end{pmatrix},\ \frac{1}{\sqrt{10}}\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix}\right\} \qquad \mathrm{Nul}(A^T):\ \{\}.$$

Problem 4. [20 points]

Consider the difference equation

$$\begin{aligned} x_{n+1} &= 2x_n - \phantom{\tfrac{1}{2}}y_n \qquad & x_0 &= 1 \\ y_{n+1} &= \tfrac{3}{2}x_n - \tfrac{1}{2}y_n \qquad & y_0 &= 2. \end{aligned}$$

a) Find a matrix A such that

$$A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.$$

b) Find the eigenvalues of A, and find corresponding eigenvectors.

c) Find a formula for $\binom{x_n}{y_n}$ in terms of n.

d) What is $\lim_{n\to\infty} \binom{x_n}{y_n}$?

Solution.

a) The matrix is $A = \begin{pmatrix} 2 & -1 \\ 3/2 & -1/2 \end{pmatrix}$.

b) The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and corresponding eigenvectors are $w_1 = \binom{1}{1}$ and $w_2 = \binom{2}{3}$.

c) We have $\binom{x_0}{y_0} = -w_1 + w_2$, so

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -A^n w_1 + A^n w_2 = -w_1 + \frac{1}{2^n} w_2 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2^n}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

d) The limit is $-w_1 = -\binom{1}{1}$.

Problem 5. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A symmetric matrix satisfying

$$S\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\4\\6\end{pmatrix} \quad \text{and} \quad S\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}.$$

b) A 2×2 matrix whose 1-eigenspace is the line $x+2y=0$ and whose 2-eigenspace is the line $x+3y=0$.

c) A 2×2 matrix that is neither invertible nor diagonalizable.

d) A 2×2 non-invertible matrix with eigenvalue $2+3i$.

e) A 2×2 matrix A that is diagonalizable over $\mathbf{R}$, such that A^2 is not diagonalizable.

Solution.

a) Does not exist: eigenvectors with different eigenvalues would have to be orthogonal.

b) This matrix satisfies

$$A=\begin{pmatrix}-2&-3\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}-2&-3\\1&1\end{pmatrix}^{-1}=\begin{pmatrix}4&6\\-1&-1\end{pmatrix}.$$

c) One example is $\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

d) Does not exist: the other eigenvalue would be $2-3i$, so 0 is not an eigenvalue.

e) Does not exist: if $A=CDC^{-1}$ then $A^2=CD^2C^{-1}$.

Problem 6.

[5 points]

Let A be an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2.$$

Which of the following can you determine from this information? (Select all that apply.)

(1) The number n.
(2) The trace of A.
(3) The determinant of A.
(4) The rank of A.
(5) Whether A is symmetric.
(6) Whether A is diagonalizable.
(7) The eigenvalues of A.
(8) The singular values of A.

Solution.

You can determine (1), (2), (3), (4), and (7).

(1) $n = \deg(p) = 4$.
(2) $\mathrm{Tr}(A)$ is the sum of the eigenvalues (with multiplicity), which is $2 + 3 + 3 = 8$.
(3) $\det(A)$ is the product of the eigenvalues (with multiplicity), which is 0.
(4) The null space is the 0-eigenspace, which has algebraic multiplicity 1, hence also geometric multiplicity 1. Therefore $\dim \mathrm{Nul}(A) = 1$, so $\mathrm{rank}(A) = 4 - 1 = 3$.
(5) You can't tell. Both of these matrices have characteristic polynomial $p(\lambda)$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(6) You can't tell. The first matrix above is diagonal, and the second is not diagonalizable.
(7) The eigenvalues are $0, 2$, and 3.
(8) You can't tell: the first matrix above has singular values 3 and 2, and the second has singular values ≈ 3.54 and 2.

Problem 7.

[10 points]

A certain diagonalizable 2×2 matrix A has eigenvalues 1 and 2, with eigenspaces drawn below.

a) Draw Ax and Ay on the diagram.

b) Draw the vector $w = \lim_{n \to \infty} A^n x / \|A^n x\|$: that is, eventually $A^n x$ points in the direction of the unit vector w. (Let's say that 1cm on your paper is one unit.)