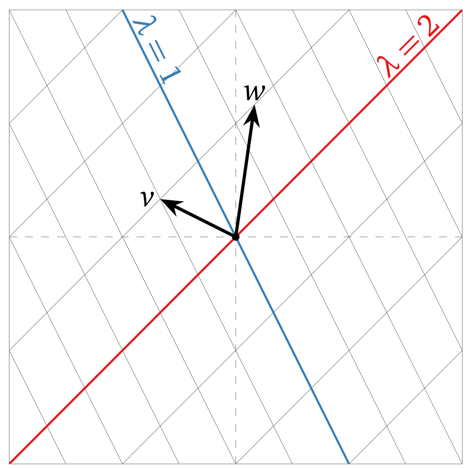
**Homework #10**

due Tuesday, October 27, at 11:59pm

**1.** A certain $2 \times 2$ matrix $A$ has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.

**a)** Draw $Av$, $A^2v$, and $Aw$.

**b)** Describe what happens to $A^n v$ as $n \to \infty$.



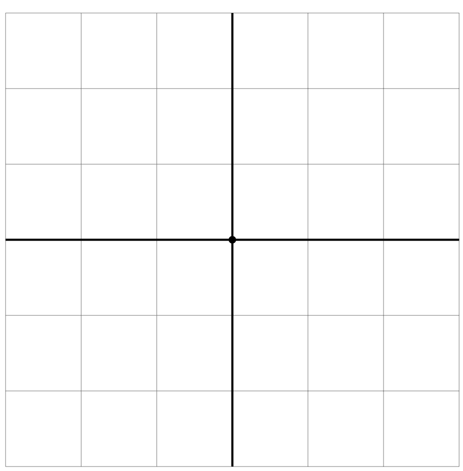

**2.** A certain diagonalizable $2 \times 2$ matrix $A$ is equal to $CDC^{-1}$, where $C$ has columns $w_1, w_2$ pictured below, and $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$.

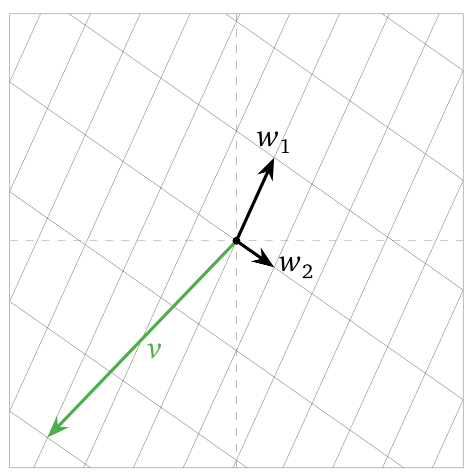
**a)** Draw $C^{-1}v$ on the left.

**b)** Draw $DC^{-1}v$ on the left.

**c)** Draw $Av = CDC^{-1}v$ on the right.

**d)** What happens to $A^n v$ as $n \to \infty$?





**3.** Compute the following complex numbers.

**a)** $(1+i)+(2-i)$ **b)** $(1+i)(2-i)$ **c)** $\overline{2-i}$ **d)** $\dfrac{1+i}{2-i}$

**e)** $|1+i|$ **f)** $2e^{2\pi i/3}$ **g)** $5e^{3\pi i}$

**4.** Express each complex number in polar coordinates $re^{i\theta}$.

**a)** $1+i$ **b)** $\dfrac{-1+i\sqrt{3}}{2}$ **c)** $-\sqrt{3}-3i$ **d)** $\dfrac{1}{1+i}$ **e)** $\left(1-i\sqrt{3}\right)^n$

**5.** For which numbers $\theta$ is $e^{i\theta}=1$? What about $-1$?

**6.** For each matrix $A$ and each vector $x$, decide if $x$ is an eigenvector of $A$, and if so, find the eigenvalue $\lambda$.

**a)** $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$, $\begin{pmatrix} i \\ 1 \end{pmatrix}$ **b)** $\begin{pmatrix} -4 & 13 & 13 \\ 2 & -2 & -4 \\ -4 & 8 & 10 \end{pmatrix}$, $\begin{pmatrix} 1+5i \\ -2i \\ 4i \end{pmatrix}$

**c)** $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ -2 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 2+i \\ 1 \\ -i \end{pmatrix}$

Careful! It is difficult to recognize by inspection if two complex vectors are (complex) scalar multiples of each other.

**7.** For each $2\times 2$ matrix $A$, **i)** compute the characteristic polynomial, **ii)** find all (real and complex) eigenvalues, and **iii)** find a basis for each eigenspace, using Problem 3 on Homework 9 when applicable. **iv)** Is the matrix diagonalizable (over the complex numbers)? If so, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A=CDC^{-1}$.

**a)** $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 5 \\ -10 & 7 \end{pmatrix}$

**8.** Diagonalize the following matrix over the complex numbers:

$$A=\begin{pmatrix} 1 & 4 & -6 \\ -6 & 7 & -22 \\ -2 & 1 & -5 \end{pmatrix}.$$

One eigenvalue is $\lambda=-1$.

**9.** A certain forest contains a population of rabbits and a population of foxes. If there are $r_n$ rabbits and $f_n$ foxes in year $n$, then

$$\begin{aligned} r_{n+1} &= 3r_n - f_n \\ f_{n+1} &= r_n + 2f_n: \end{aligned}$$

in other words, each rabbit produces three baby rabbits on average, but there is some loss due to predation by foxes; each fox produces two babies on average, but this is increased with ample prey.

**a)** Let $v_n = \binom{r_n}{f_n}$. Find a matrix $A$ such that $v_{n+1} = Av_n$.

**b)** Find an eigenbasis of $A$. (The eigenvectors and eigenvalues will be complex.) [**Hint:** Part **c)** will be easier if you choose the eigenvectors with first coordinate equal to 1.]

**c)** Suppose that $r_0 = 2$ and $f_0 = 1$. Find closed formulas for $r_n$ and $f_n$. Find a formula for $r_n$ involving only real numbers. (This latter formula can involve an arctan.)

**d)** In this model, the populations do not stabilize. How many years will it take for the foxes to eat all of the rabbits?

In general, any $2 \times 2$ difference equation with a complex eigenvalue will exhibit oscillation centered at zero. This phenomenon can be described explicitly, but is beyond the scope of this course.

**10.** Solve the following initial value problems. Your solutions should involve only real numbers.

**a)** $\begin{cases} u_1' = u_1 - 2u_2 & u_1(0) = -3 \\ u_2' = u_1 + 4u_2 & u_2(0) = \phantom{-}2 \end{cases}$ **b)** $\begin{cases} u_1' = 3u_1 - \phantom{2}u_2 & u_1(0) = 4 \\ u_2' = \phantom{3}u_1 + 2u_2 & u_2(0) = 2 \end{cases}$

**11.** **a)** Let $A$ be an $n \times n$ matrix. Prove that $\lambda$ is an eigenvalue of $A$ with geometric multiplicity $n$ if and only if $A = \lambda I_n$.

**b)** Find a non-diagonal $2 \times 2$ matrix such that 1 is an eigenvalue with algebraic multiplicity 2.

**12.** Find examples of real $2 \times 2$ matrices $A$ with the following properties.

**a)** $A$ is invertible and diagonalizable over the real numbers.

**b)** $A$ is invertible but not diagonalizable over the complex numbers.

**c)** $A$ is diagonalizable over the real numbers but not invertible.

**d)** $A$ is neither invertible nor diagonalizable over the complex numbers.

This shows that *invertibility and diagonalizability have nothing to do with each other*.

**13.** Let $A$ be an $n \times n$ matrix.

**a)** Show that the product of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\det(A)$.

**b)** [**Optional**] Show that the sum of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\mathrm{Tr}(A)$.

(Both of these are identities involving the characteristic polynomial of $A$.)

**14.** Let $L$ be a line in $\mathbf{R}^3$, let $P_L$ be orthogonal projection onto $L$, and let $R_L = I_3 - 2P_L$ be the reflection over the orthogonal plane.

**a)** Prove that there exists an invertible $3 \times 3$ matrix $C$ such that

$$P_L = C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} C^{-1}.$$

Use this to show that the characteristic polynomial of $P_L$ is $-\lambda^2(\lambda - 1)$.

**b)** Prove that

$$R_L = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} C^{-1}$$

for the same matrix $C$ of part **a)**. Use this to show that the characteristic polynomial of $R_L$ is $-(\lambda-1)^2(\lambda+1)$ and that $\det(R_L) = -1$. (Compare Problem 10 on Homework 8.)

**15.** For each matrix in Problem 5(a)–(c) on Homework 9, compute the algebraic and geometric multiplicity of each eigenvalue. What does your answer say about diagonalizability? **Optional:** do (d)–(g) as well.

**16.** Give an example of each of the following, or explain why no such example exists. All matrices should have real entries.

**a)** A $3 \times 3$ matrix with eigenvalues $0, 1, 2$, and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

**b)** A $4 \times 4$ matrix having eigenvalue 2 with algebraic multiplicity 2 and geometric multiplicity 3.

**c)** A $3 \times 3$ matrix with one complex eigenvalue and two real eigenvalues.

**d)** A $2 \times 2$ matrix $A$ such that $A^2$ is diagonalizable over the real numbers but $A$ is not diagonalizable, even over the complex numbers.
[**Hint:** try a nonzero matrix $A$ such that $A^2 = 0$.]

**17.** Decide if each statement is true or false, and explain why.

**a)** If $A$ and $B$ are diagonalizable $n \times n$ matrices, then so is $AB$.

**b)** An $n \times n$ matrix with $n$ (different) eigenvalues is diagonalizable.

**c)** An $n \times n$ matrix is diagonalizable if it has $n$ eigenvalues, counted with algebraic multiplicity.

**d)** Any $2 \times 2$ real matrix with a (non-real) complex eigenvalue is diagonalizable over the complex numbers.

**e)** Any $3 \times 3$ real matrix with a (non-real) complex eigenvalue is diagonalizable over the complex numbers.

**f)** Any $4 \times 4$ real matrix with a (non-real) complex eigenvalue is diagonalizable over the complex numbers.

**g)** Any $2 \times 2$ real matrix has a real eigenvalue.

**h)** Any $3 \times 3$ real matrix has a real eigenvalue.

**i)** Any $n \times n$ matrix has a (real or complex) eigenvalue.

**j)** If the characteristic polynomial of $A$ is $-(\lambda^3 - 1) = -(\lambda^2 + \lambda + 1)(\lambda - 1)$, then the 1-eigenspace of $A$ is a line.