**Homework #11**

due **Thursday**, November 5, at 11:59pm

**1.** For each symmetric matrix $S$, find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $S = QDQ^T$.

$$\textbf{a)} \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix} \qquad \textbf{e)} \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$$

The eigenvalues in **d)** are $3, 6, 9$ and in **e)** are $-9, 9$.

**2.** For each matrix $S$ of Problem 1, decide if $S$ is positive-semidefinite, and if so, compute its positive-semidefinite square root $\sqrt{S} = Q\sqrt{D}Q^T$. Verify that $(\sqrt{S})^2 = S$.

**Remark:** Since $\sqrt{S}$ is also symmetric, we have $S = \sqrt{S}^T\sqrt{S}$, so this is another way to factorize a positive-semidefinite matrix as $A^TA$.

**3.** Consider the matrix

$$S = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

of Problem 1(d). Write $S$ in the form $\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T$ for numbers $\lambda_1, \lambda_2, \lambda_3$ and orthonormal vectors $q_1, q_2, q_3$.
[**Hint:** Use the columns of $Q$. Why does this work?]

**4.** Find *all possible* orthogonal diagonalizations

$$\frac{1}{5}\begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} = QDQ^T.$$

**5.** Suppose that $A$ is a square matrix such that $A^k = 0$ for some $k > 0$.

**a)** Show that 0 is the only eigenvalue of $A$.

**b)** Show that $A = 0$ if it is symmetric.

**6.** Let $S$ be a symmetric orthogonal $2 \times 2$ matrix.

**a)** Show that $S = \pm I_2$ if it has only one eigenvalue.

**b)** Suppose that $S$ has two eigenvalues. Show that $S$ is the matrix for the reflection over a line $L$ in $\mathbf{R}^2$. (Recall that the reflection over a line $L$ is given by $R_L = I_2 - 2P_{L^\perp}$.)
[**Hint:** Write $S$ as $\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$, and use the projection formula to write $I_2$ and $P_{L^\perp}$ in this form as well.]

**7.** **a)** Let $S$ be a diagonalizable (over $\mathbf{R}$) $n \times n$ matrix with orthogonal eigenspaces: that is, eigenspaces with different eigenvalues are orthogonal subspaces. Prove that $S$ is symmetric.

[**Hint:** choose *orthonormal* bases for each eigenspace.]

**b)** Let $S$ be a matrix that can be written in the form

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

for some vectors $q_1, q_2, \ldots, q_n$. Prove that $S$ is symmetric.

**c)** Let $V$ be a subspace of $\mathbf{R}^n$, and let $P_V$ be the projection matrix onto $V$. Use **a)** or **b)** to prove that $P_V$ is symmetric. (Compare Problem 8 on Homework 6.)

**8.** For each symmetric matrix $S$, decide if $S$ is positive-definite. If so, find its $LDL^T$ and Cholesky decompositions. Do not compute any eigenvalues!

$$\textbf{a)} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 14 & 8 \\ 1 & 3 & 8 & 9 \end{pmatrix} \qquad \textbf{e)} \begin{pmatrix} -1 & 2 & 3 & -2 \\ 2 & -3 & -8 & 4 \\ 3 & -8 & -4 & 6 \\ -2 & 4 & 6 & -1 \end{pmatrix}$$

**9.** For which matrices $A$ is $S = A^T A$ positive-definite? If $S$ is not positive-definite, find a vector $x$ such that $x^T S x = 0$. In any case, do not compute $S$!

$$\textbf{a)} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

**10.** **a)** For each symmetric matrix $S$, compute the associated quadratic form $q(x) = x^T S x$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

**b)** Let $A$ be a square matrix and let $S = \frac{1}{2}(A + A^T)$. Show that $S$ is symmetric and that $x^T A x = x^T S x$. (This is why we only consider symmetric matrices when studying quadratic forms.)

**11.** For each quadratic form $q(x_1, x_2)$, **i)** write $q(x)$ in the form $x^T S x$ for a symmetric matrix $S$, **ii)** find coordinates $y_1, y_2$ such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2$, **iii)** draw the solutions of $q(x_1, x_2) = 1$, being sure to draw the shortest and longest solutions, and **iv)** find the maximum and maximum values of $q(x_1, x_2)$ subject to the constraint

$x_1^2 + x_2^2 = 1$, and at which points $(x_1, x_2)$ these values are attained.

**a)** $q(x_1, x_2) = 14x_1^2 + 4x_1x_2 + 11x_2^2$ **b)** $q(x_1, x_2) = \frac{1}{10}(21x_1^2 - 6x_1x_2 + 29x_2^2)$

**c)** $q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2$

[**Hint:** An equation of the form $(x_1/r_1)^2 - (x_2/r_2)^2 = 1$ defines a hyperbola.]

**12.** For the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3,$$

find coordinates $y_1, y_2, y_3$ such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$, and find the maximum and minimum values of $q(x_1, x_2, x_3)$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$, along with the points $(x_1, x_2, x_3)$ at which these values are attained.

**13.** **a)** If $S$ is positive-definite and $C$ is invertible, show that $CSC^T$ is positive-definite.

**b)** If $S$ and $T$ are positive-definite, show that $S + T$ is positive-definite.

**c)** If $S$ is positive-definite, show that $S$ is invertible and that $S^{-1}$ is positive-definite.

[**Hint:** For **a)** and **b)** use the positive-energy characterization of positive-definiteness; for **c)** use the positive-eigenvalue characterization.]

**14.** Let $S$ be a positive-definite matrix.

**a)** Show that the diagonal entries of $S$ are positive.
[**Hint:** compute $e_i^T S e_i$.]

**b)** Show that the diagonal entries of $S$ are all greater than or equal to the smallest eigenvalue of $S$.
[**Hint:** if not, apply **a)** to $S - aI_n$ for a diagonal entry $a$ that is smaller than all eigenvalues.]

**15.** Decide if each statement is true or false, and explain why. All matrices are real.

**a)** A symmetric matrix is diagonalizable.

**b)** If $A$ is any matrix then $A^T A$ is positive-semidefinite.

**c)** A symmetric matrix with positive determinant is positive-definite.

**d)** A positive-definite matrix has the form $A^T A$ for a matrix $A$ with full column rank.

**e)** If $A = CDC^{-1}$ for a diagonal matrix $D$ and a non-orthogonal invertible matrix $C$, then $A$ is not symmetric.

**f)** The only positive-definite projection matrix is the identity.

**g)** All eigenvalues of a positive-definite symmetric matrix are positive real numbers.