Homework #12

due **Thursday**, November 12, at 11:59pm

Note on the first two problems: I'm sorry these are so tedious. I made an effort to create the nicest SVD problems that I could while still exhibiting all of the intricacies of these computations, but I'm afraid these things are inherently messy. So don't be surprised if your answers have a $\sqrt{78}$ and a $\sqrt{165}$ in them. I recommend verifying your answers numerically, with linalg.js or otherwise.

1. For each matrix A, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

$$\textbf{a)} \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} \qquad \textbf{e)} \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

[**Hint:** one of the singular values in **e)** is 12.]

2. For each matrix A of Problem 1, find the singular value decomposition in the matrix form

$$A = U\Sigma V^T.$$

3. For each matrix A of Problem 1, write down orthonormal bases for all four fundamental subspaces. (This can be read off from your answers to Problem 2.)

4. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 1(a). Let σ_1, σ_2 be the singular values of A.

a) Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

b) Find an orthonormal eigenbasis $\{v_1, v_2\}$ of $A^T A$ such that $A^T A v_i = \sigma_i^2 v_i$ and an orthonormal eigenbasis $\{u_1, u_2\}$ of AA^T such that $AA^T u_i = \sigma_i^2 u_i$, such that A is *not* equal to $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

[**Hint:** The condition $Av_i = \sigma_i u_i$ is not automatic!]

5. Find the matrix A satisfying

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

and write the SVD of A in vector form.

[**Hint:** Start by finding the SVD.]

6. Let A be a matrix with nonzero orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of A in vector form.

7. **a)** Let A be an invertible $n \times n$ matrix. Show that the product of the singular values of A equals the absolute value of the product of the (real and complex) eigenvalues of A (counted with algebraic multiplicity).

[**Hint:** Both equal $|\det(A)|$.]

b) Find an example of a 2×2 matrix A with distinct positive eigenvalues that are not equal to any of the singular values of A.

[**Hint:** One of the matrices in Problem 1 works.]

8. Let S be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$.

a) Show that the singular values of S are $|\lambda_1|, \ldots, |\lambda_r|$. In particular, $\text{rank}(S) = r$.

b) Suppose that $S = QDQ^T$, where Q is orthogonal and D is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Show that S has a singular value decomposition of the form $U\Sigma Q^T$ (i.e., $V = Q$). How is Σ related to D? How is U related to Q?

c) Show that $S = QDQ^T$ is a singular value decomposition if and only if S is positive-semidefinite.

9. **a)** Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use Problem 11 on Homework 9.]

Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

b) Show that all singular values of an orthogonal matrix are equal to 1.

10. Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A.

a) Show that A and R have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.

b) What is the relationship between the left singular vectors of A and R?

11. Let A be a matrix with first singular value σ_1 and first right singular vector v_1.

a) Show that the maximum value of $\|Ax\|$ subject to $\|x\| = 1$ is the same as the maximum value of $\|Ax\|/\|x\|$ subject to $x \neq 0$.

b) Show that $\|Ax\|/\|x\|$ is maximized at $x = v_1$, with maximum value σ_1.

[**Hint:** How do you maximize $\|Ax\|^2 = x^T(A^TA)x$ for $\|x\| = 1$?]

c) Suppose now that A is square and λ is an eigenvalue of A. Show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

This shows that *the largest singular value is at least as big as the largest eigenvalue.*

Remark: The maximum value of $\|Ax\|/\|x\|$ for $x \neq 0$ is called the *norm* of A and is denoted $\|A\|$.

12. Let A be a square, invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$.

a) Show that A^{-1} has the same singular vectors as A^T, with singular values $\sigma_n^{-1} \geq \cdots \geq \sigma_1^{-1}$.

[**Hint:** Invert $A = U\Sigma V^T$.]

b) Let λ be an eigenvalue of A. Use Problem 11(c) and **a)** to show that $\sigma_n \leq |\lambda|$.

13. **a)** Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A'? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are not*. This is another advantage of the SVD.

14. Decide if each statement is true or false, and explain why.

a) The left singular vectors of A are eigenvectors of A^TA and the right singular vectors are eigenvectors of AA^T.

b) For any matrix A, the matrices AA^T and A^TA have the same nonzero eigenvalues.

c) If S is symmetric, then the nonzero eigenvalues of S are its singular values.

d) If A does not have full column rank, then 0 is a singular value of A.

e) Suppose that A is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

f) The right singular vectors of A are orthogonal to Nul(A).