Homework #2

due Tuesday, September 1, at 11:59pm

1. Which of the following matrices are not in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

2. Describe all possible nonzero 2×2 matrices in RREF.

3. Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer.

$$\textbf{a)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \textbf{b)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) \quad \textbf{c)} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \quad \textbf{e)} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right) \quad \textbf{f)} \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

4. Use Gaussian elimination and back-substitution to solve

$$\textbf{a)} \begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ x_2 + 2x_3 = 3 \end{cases} \qquad \textbf{b)} \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 8x_3 = 1. \end{cases}$$

What happens if you replace 8 by 9 in (b)?

5. Three planes can fail to have an intersection point, even if no planes are parallel. Consider the two planes $A : x + y + z = 0$ and $B : x - 2y - z = 1$. Use the tool here `https://technology.cpm.org/general/3dgraph/` to visualize these two planes, then answer the following questions:

a) What is the shape of the intersection $A \cap B$ of the two?

b) Use the equations of A and B to construct a third plane C whose intersection with the two is exactly the same as $A \cap B$. That is, $A \cap B \cap C = A \cap B$ [**Hint:** how can you create a system of three equations with fewer than three pivots?]

c) Find a fourth plane D such that $A \cap D$, and $B \cap D$ are both non-empty, but $A \cap B \cap D$ is empty. That is, D should intersect both A and B, but the three should never meet. [**Hint:** make the system inconsistent!]

For both the last two parts, I strongly suggest you use the tool linked above to draw the planes and see your answers!

6. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.

7. Find values of a and b such that the following system is **a)** inconsistent and **b)** consistent.

$$\begin{aligned} 2x + ay &= 4 \\ x - y &= b \end{aligned}$$

8. In the matrix equation $Ax = b$, suppose that A has a pivot position in every *row*. Explain why the linear system $Ax = b$ is consistent. [**Hint:** What happens when you do back-substitution?]

9. In the matrix equation $Ax = b$, suppose that A has a pivot position in every *column*. Explain why the linear system $Ax = b$ has at most one solution. [**Hint:** What happens when you do back-substitution?]

10. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:

a) $R_2 \mathrel{+}= 2R_1$

b) $R_1 \mathrel{-}= \frac{1}{2}R_3$

c) $R_3 \times= 2$

d) $R_3 \div= 2$

e) $R_1 \longleftrightarrow R_3$

f) $R_1 \longleftrightarrow R_2$

11. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

$$\textbf{a)}\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{b)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{c)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \textbf{e)}\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

12. For each elementary matrix in the previous problem, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row op}} R_2 \mathrel{+}= R_1 \xrightarrow{\text{undo}} R_2 \mathrel{-}= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

13. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \underline{\qquad\qquad} A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \underline{\qquad\qquad} U$$

d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6.

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \underline{\qquad\qquad\qquad}.$$

14. Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

15. Compute the inverse of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why.

a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

16. Consider the linear system

$$\begin{aligned} x_1 + x_2 \phantom{{}+2x_3} &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the previous problem to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3.

17. Decide if each statement is true or false, and explain why.

a) If A and B are invertible $n \times n$ matrices, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

b) An $n \times n$ matrix with n pivots is invertible.

c) An invertible $n \times n$ matrix has n pivots.

18. Suppose that

$$A\begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad A\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad A\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

What is A^{-1}?

19. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition.

a)

$$\begin{pmatrix}3 & 2 & 7\\ -6 & -5 & -10\\ -3 & 0 & -13\end{pmatrix} x = \begin{pmatrix}14\\ -26\\ -16\end{pmatrix}$$

$$\begin{pmatrix}3 & 2 & 7\\ -6 & -5 & -10\\ -3 & 0 & -13\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & -2 & 1\end{pmatrix}\begin{pmatrix}3 & 2 & 7\\ 0 & -1 & 4\\ 0 & 0 & 2\end{pmatrix}$$

b)

$$\begin{pmatrix}2 & 4 & -3 & 2\\ -2 & -7 & 7 & -7\\ 4 & 17 & -17 & 19\\ 2 & 4 & -5 & 1\end{pmatrix} x = \begin{pmatrix}3\\ -4\\ 10\\ 0\end{pmatrix}$$

$$\begin{pmatrix}2 & 4 & -3 & 2\\ -2 & -7 & 7 & -7\\ 4 & 17 & -17 & 19\\ 2 & 4 & -5 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 2 & -3 & 1 & 0\\ 1 & 0 & -2 & 1\end{pmatrix}\begin{pmatrix}2 & 4 & -3 & 2\\ 0 & -3 & 4 & -5\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

c)

$$\begin{pmatrix} 6 & 1 & 16 \\ 2 & 3 & -1 \\ 4 & 4 & 3 \end{pmatrix} x = \begin{pmatrix} -21 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1 & 16 \\ 2 & 3 & -1 \\ 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

20. Compute the $A = LU$ decomposition of the following matrices.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

21. Compute a $PA = LU$ decomposition for each of the following matrices, using maximal partial pivoting.

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$