Homework #3

due Tuesday, September 8, at 11:59pm

1. Express each system of linear equations as a vector equation. For example,

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 - x_2 &= 4 \end{aligned} \quad \rightsquigarrow \quad x_1\begin{pmatrix}1\\-1\end{pmatrix} + x_2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}.$$

a) $$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$$

b) $$\begin{pmatrix}3 & -5\\ 2 & 4\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

c) $$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2\\ 0 & 3 & -1 & -2 & 4\\ 1 & -3 & -4 & -3 & 2\\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$$

2. For each matrix A and vector b, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \rightsquigarrow \quad x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

a) $$A = \begin{pmatrix}2 & 1 & 1 & 4\\ 4 & 2 & 1 & 7\end{pmatrix} \qquad b = \begin{pmatrix}1\\1\end{pmatrix}$$

b) $$A = \begin{pmatrix}2 & 2 & -1\\ -4 & -5 & 5\\ 6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\49\end{pmatrix}$$

c) $$A = \begin{pmatrix}2 & 2 & -1\\ -4 & -5 & 5\\ 6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\48\end{pmatrix}$$

d) $$A = \begin{pmatrix}1 & 2 & 3 & -1 & 1\\ -2 & -4 & -5 & 4 & 1\\ 1 & 2 & 2 & -3 & -1\\ -3 & -6 & -7 & 7 & 6\end{pmatrix} \qquad b = \begin{pmatrix}2\\4\\-6\\10\end{pmatrix}$$

e) $$A = \begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix} \qquad b = \begin{pmatrix}2\\5\\4\end{pmatrix}$$

3. For each matrix A and vector b in Problem 2, find the parametric *vector* form of the general solution of $Ax = b$, and express the solution set in the form

$$\mathrm{Span}\{???\} + p$$

for some vector p. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xrightarrow{\text{soln set}} \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

4. For each matrix A in Problem 2, determine if the homogeneous equation $Ax = 0$ has a nontrivial solution. If it does, write the solution set of $Ax = 0$ as a span.

5. Find values of a and b such that the following system has **a)** zero **b)** one **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - \ \ y &= b \end{aligned}$$

6. The equation $x + 2y = z$ determines a plane in $\mathbf{R}^3$. (This is an *implicit equation* for the plane).

a) What is the coefficient matrix A for this system?

b) Which are the free variables?

c) Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.

d) Do the same for the plane defined by $2y = z$. What is different?

7. Give examples of matrices A in reduced row echelon form for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

8. **a)** Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ in $\mathrm{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}\right\}$?

If so, express $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$.

[**Hint:** compare Problem 2.]

b) Find a vector that is *not* in the span of the columns of $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$.

9. Describe and compare (geometrically) the solution sets of the following systems:

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases} \qquad \begin{cases} 2x_1 + x_2 + x_3 = 2 \\ 4x_1 + 2x_2 + x_3 = 0. \end{cases}$$

10. Decide if each statement is true or false, and explain why.

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An $m \times n$ matrix has at most m pivots.

d) A wide matrix (more columns than rows) must have a free variable.

e) If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.

11. Let A be a 4×5 matrix with four pivots. Suppose that

$$(\text{column } 1) + 2(\text{column } 3) - (\text{column } 4) = 0.$$

a) Find a nonzero solution of $Ax = 0$.

b) Which is the free variable?

[**Hint:** the solutions of $Ax = 0$ are unchanged by Gauss–Jordan elimination.]

12. When is the following system consistent?

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= b_1 \\ -4x_1 - 5x_2 + 5x_3 &= b_2 \\ 6x_1 + x_2 + 12x_3 &= b_3 \end{aligned}$$

Your answer should be a single linear equation in b_1, b_2, b_3. Explain the relationship between this equation and

$$\text{Span}\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}.$$

13. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix} 1+t \\ 2-t \\ t \end{pmatrix} \qquad t \in \mathbf{R}.$$

14. Suppose that A is a 3×3 matrix and b is a vector such that $Ax = b$ is a line in $\mathbf{R}^3$. How many pivots does A have?

15. Suppose that A is a 2×3 matrix such that

$$A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

a) Find two different solutions of $Ax = 0$.

b) Find two more solutions of $Ax = \binom{-1}{1}$.

16. Suppose that $Ax = \binom{1}{-1}$ has infinitely many solutions. Is it possible for $Ax = \binom{2}{0}$ to have exactly one solution? How about zero solutions?

17. Suppose that $Ax = b$ is consistent. Explain why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

18. List five different vectors in $\mathrm{Span}\{x, y\}$, where

$$x = \begin{pmatrix}1\\2\\-3\end{pmatrix} \qquad y = \begin{pmatrix}-2\\3\\-1\end{pmatrix}.$$

19. Give geometric descriptions of the following spans.

a) $\mathrm{Span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$ **b)** $\mathrm{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **c)** $\mathrm{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\-6\end{pmatrix}\right\}$

d) $\mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ **e)** $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

[**Hint:** for **d)**, compare Problem 12.]

20. Decide if each statement is true or false, and explain why.

a) A vector b is a linear combination of the columns of A if and only if $Ax = b$ has a solution.

b) If $Ax = b$ has no solutions, then A does not have a pivot in every row.

c) The zero vector is contained in every span.

d) The matrix equation $Ax = 0$ can be consistent or inconsistent, depending on what A is.

e) If the zero vector is a solution of a system of equations, then the system is homogeneous.

f) If $Ax = b$ has a unique solution, then A has a pivot in every column.

g) If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

h) It is possible for $Ax = b$ to have exactly 13 solutions.