Homework #4

due **Thursday**, September 17, at 11:59pm

1. **a)** For the following matrix A, compute $A^T, A^{-1}, (A^T)^{-1}$, and $(A^{-1})^T$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Which two will always be equal?

b) For the following matrices A and B, compute $AB, (AB)^T, A^T B^T$, and $B^T A^T$:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Which two will always be equal?

2. **a)** Find a nonzero 2×2 matrix such that $A^2 = 0$.

b) Show that $A^T A = 0$ is only possible when $A = 0$.

3. Let Q be an $n \times n$ matrix such that $Q^T Q = I_n$ (so $Q^T = Q^{-1}$).

a) Show that the columns of Q are unit vectors.

b) Show that the columns of Q are orthogonal to each other.

c) Find all 2×2 matrices Q such that $Q^T Q = I_2$.

Such a matrix Q is called *orthogonal*.

4. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is symmetric, then A^{-1} is symmetric.

d) If A is any matrix, then $A^T A$ is symmetric.

5. Let R be the reduced row echelon form of a matrix A. Explain why the following quantities are all equal to the rank of A:

a) The number of pivots of A.

b) The number of nonzero rows of R.

c) The number of columns of A minus the number of free variables.

6. Let A be a matrix such that $\mathrm{Nul}(A) = \mathrm{Span}\{(1,2,3,4)\}$. What is the rank of A, and why?

7. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

$$\textbf{a)}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

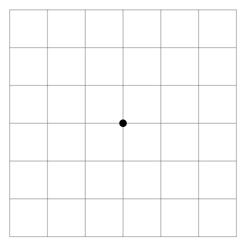
$$\textbf{c)}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \textbf{d)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** Compare Problem 4 on Homework 3.]

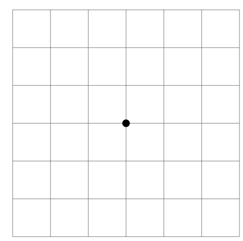
8. Draw pictures of the null space and the column space of the following matrices. Be precise!

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul(A)

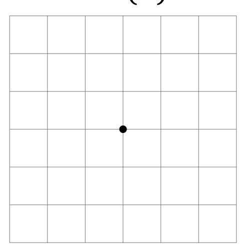


Col(A)

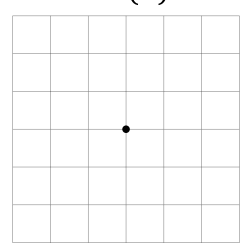


b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

Nul(A)

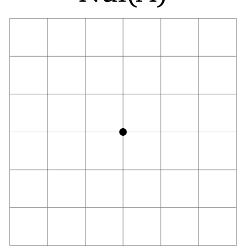


Col(A)

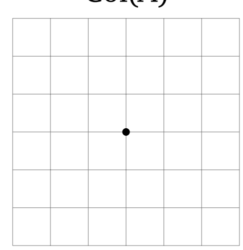


c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

Nul(A)



Col(A)



9. Give examples of subsets V of $\mathbf{R}^2$ such that:

a) V is closed under addition and contains 0, but is not closed under scalar multiplication.

b) V is is closed under scalar multiplication and contains 0, but is not closed under addition.

c) V is closed under addition and scalar multiplication, but does not contain 0.

Therefore, none of these conditions is redundant.

10. Which of the following subsets of $\mathbf{R}^3$ are subspaces? If it is not a subspace, why not? If it is, write it as the column space or null space of some matrix.

a) The plane $\{(x,y,x) \colon x,y \in \mathbf{R}\}$.

b) The plane $\{(x,y,1) \colon x,y \in \mathbf{R}\}$.

c) The set consisting of all vectors (x,y,z) such that $xy = 0$.

d) The set consisting of all vectors (x,y,z) such that $x \leq y$.

e) The span of $(1,2,3)$ and $(2,1,-3)$.

f) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

g) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 1. \end{cases}$

11. Give a geometric description of the following column spaces (line, plane, ...).

$$\textbf{a) } \mathrm{Col}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \textbf{b) } \mathrm{Col}\begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} \qquad \textbf{c) } \mathrm{Col}\begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & -6 \end{pmatrix}$$

$$\textbf{d) } \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad \textbf{e) } \mathrm{Col}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** Compare Problem 19 on Homework 3.]

12. **a)** Explain why $\mathrm{Col}(AB)$ is contained in $\mathrm{Col}(A)$.

b) Give an example where $\mathrm{Col}(AB) \neq \mathrm{Col}(A)$. Can you take $A = B$?
[**Hint:** use Problem 2(a).]

13. **a)** Explain why $\mathrm{Nul}(AB)$ contains $\mathrm{Nul}(B)$.

b) Give an example where $\mathrm{Nul}(AB) \neq \mathrm{Nul}(B)$. Can you take $A = B$?
[**Hint:** use Problem 2(a).]

14. If $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$, then $AB =$ __________.

15. Find a 2×2 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$. What is the rank of such a matrix?
[**Hint:** use Problem 2(a).]

16. Find a matrix A such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}\right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}.$$

What is the rank of A?

17. Decide if each statement is true or false, and explain why.

a) The column space of an $m \times n$ matrix with m pivots is $\mathbf{R}^m$.

b) The null space of an $m \times n$ matrix with n pivots is $\mathbf{R}^n$.

c) If $\mathrm{Col}(A) = \{0\}$, then A is the zero matrix.

d) If $\mathrm{Nul}(A) = \{0\}$, then A has full column rank.

e) If A has full row rank, then $\mathrm{Col}(A) = \mathbf{R}^m$.

f) The column space of $2A$ equals the column space of A.

g) The null space of $A + B$ contains the null space of A.

h) If U is an echelon form of A, then $\mathrm{Nul}(U) = \mathrm{Nul}(A)$.

i) If U is an echelon form of A, then $\mathrm{Col}(U) = \mathrm{Col}(A)$.

18. **a)** Give an example of a 3×3 matrix A such that $\mathrm{Col}(A)$ contains $(1, 2, 3)$ and $(1, 0, -1)$, but $\mathrm{Col}(A) \neq \mathbf{R}^3$. What is the rank of A?

b) Give an example of a 3×3 matrix A, with no zero entries, such that $\mathrm{Col}(A)$ is the line through $(1, 1, 1)$. What is the rank of A?

19. For the following matrix A, compute the reduced row echelon form of A and of A^T. Do they have the same free variables? Do they have the same rank?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{pmatrix}$$

20. For each matrix A, verify that $\mathrm{rank}(A) = 1$, and find vectors u, v such that $A = uv^T$. (In general, a matrix has rank 1 if and only if it is equal to a column vector times a row vector.)

$$\textbf{a)}\ A = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix} \qquad \textbf{b)}\ A = \begin{pmatrix} 2 & 1 & -1 & 4 \\ -2 & -1 & 1 & -4 \\ 2 & 1 & -1 & 4 \end{pmatrix}$$