Homework #5

due Tuesday, September 22, at 11:59pm

1. Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear dependence relation.

$$\textbf{a)}\ \left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\} \qquad \textbf{b)}\ \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\} \qquad \textbf{c)}\ \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

$$\textbf{d)}\ \left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix}, \begin{pmatrix}2\\-4\\2\\-6\end{pmatrix}, \begin{pmatrix}3\\-5\\2\\-7\end{pmatrix}, \begin{pmatrix}-1\\4\\-3\\7\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\} \qquad \textbf{e)}\ \left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$$

Which sets do you know are linearly dependent without doing any work?

2. **a)** For each set in Problem 1, find a basis for the span of the vectors.

b) For each set in Problem 1, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for **a)**.

c) What is the dimension of each of these spans?

3. Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V, and set

$$v_1 = w_2 + w_3 \qquad v_2 = w_1 + w_3 \qquad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V.

4. Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$ is the number of columns of A, that $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T)$ is the number of rows, and that $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.

[**Hint:** Augment with the $m \times m$ identity matrix so you only have to do Gauss–Jordan elimination once.]

$$\textbf{a)}\ \begin{pmatrix}2 & 1 & 1 & 4\\4 & 2 & 1 & 7\end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix}1 & 0 & 1\\1 & 1 & 2\\0 & 1 & 1\\0 & 1 & 1\end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix}1 & 2 & 3 & -1 & 1\\-2 & -4 & -5 & 4 & 1\\1 & 2 & 2 & -3 & -1\\-3 & -6 & -7 & 7 & 6\end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{pmatrix}$$

5. Consider the matrix of Problem 4(b):

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space?

6. Suppose that A is an invertible 4×4 matrix. Find bases for its four fundamental subspaces.

7. Find bases for the following subspaces.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

8. Let A be a 3×3 matrix with rank 2. Explain why A^2 is not the zero matrix.
[**Hint:** Compare Problem 14 on Homework 4.]

9. Let A be a 9×4 matrix of rank 3. What are the dimensions of its four fundamental subspaces?

10. If the left null space of a 5×4 matrix A has dimension 3, what is the rank of A?

11. Let V be a 4-dimensional subspace of $\mathbf{R}^5$.

a) Explain why every basis for V can be extended to a basis for $\mathbf{R}^5$ by adding one more vector.

b) Find an example of a 4-dimensional subspace V of $\mathbf{R}^5$ and a basis for $\mathbf{R}^5$ that cannot be reduced to a basis for V by removing one vector.

12. Find an example of a matrix with the required properties, or explain why no such exists.

a) The column space contains $(1, 2, 3)$ and $(4, 5, 6)$, and the row space contains $(1, 2)$ and $(2, 3)$.

b) The column space has basis $\{(1, 2, 3)\}$, and the null space has basis $\{(3, 2, 1)\}$.

c) The dimension of the null space is one greater than the dimension of the left null space.

d) A 3×5 matrix whose row space equals its null space.

13. **a)** Show that $\text{rank}(AB) \leq \text{rank}(A)$. [**Hint:** Compare Problem 12 on Homework 4.]

b) Show that $\text{rank}(AB) \leq \text{rank}(B)$. [**Hint:** Take transposes.]

14. This problem explains why we only consider *square* matrices when we discuss invertibility.

a) Show that a tall matrix A (more rows than columns) does not have a right inverse, i.e., there is no matrix B such that $AB = I_m$.

b) Show that a wide matrix A (more columns than rows) does not have a left inverse, i.e., there is no matrix B such that $BA = I_n$.

[**Hint:** compare Problem 13.]

15. Decide if each statement is true or false, and explain why.

a) If $v_1, v_2, \ldots, v_n$ are linearly independent vectors, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension n.

b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of A are linearly independent.

c) If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.

d) If v_3 is not a linear combination of v_1 and v_2, then $\{v_1, v_2, v_3\}$ is linearly independent.

e) If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector x.

f) The set $\{0\}$ is linearly independent.

g) If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.

h) The columns of any 4×5 matrix are linearly dependent.

i) If $Ax = \binom{1}{2}$ has only one solution, then the columns of A are linearly independent.

j) If $\text{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.

k) A and A^T have the same number of pivots.