Homework #6

due Tuesday, September 29, at 11:59pm

1. Compute a basis for the orthogonal complement of each the following subspaces.

$$\textbf{a)}\ \mathrm{Col}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix} \qquad \textbf{b)}\ \mathrm{Nul}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix} \qquad \textbf{c)}\ \mathrm{Row}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$$

$$\textbf{d)}\ \mathrm{Nul}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \qquad \textbf{e)}\ \mathrm{Span}\left\{\begin{pmatrix}3\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\} \qquad \textbf{f)}\ \mathrm{Col}\begin{pmatrix}1&2&0\\2&1&0\end{pmatrix}$$

[**Hint:** solving **a)**–**d)** requires only one Gauss-Jordan elimination, and **f)** doesn't require any work.]

2. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix}0&6&8\\\frac{1}{2}&0&0\\0&\frac{1}{2}&0\end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

g) The line $\{(t, -t, t) : t \in \mathbf{R}\}$.

[**Hint:** Compare Problem 7 on Homework 5.]

3. For each pair of vectors v and w, draw Span$\{v\}$, and compute and draw the projection p of w onto Span$\{v\}$.

$$\textbf{a)}\ v = \begin{pmatrix}1\\0\end{pmatrix},\ w = \begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix} \qquad \textbf{b)}\ v = \begin{pmatrix}-1\\1\end{pmatrix},\ w = \begin{pmatrix}1\\1\end{pmatrix}$$

4. For each vector v in Problem 3, compute the matrix P_V for projection onto $V =$ Span$\{v\}$ using the formula $P_V = vv^T/v \cdot v$. Verify that $P_V^2 = P_V$, and that $P_V w$ is equal to the projection you computed before.

5. For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by solving a normal equation $A^TAx = A^Tb$, and find the distance from b to V.

a) $V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$

b) $V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$

c) $V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$

6. For each subspace V, compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to V.

$$\textbf{a) } V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} \qquad \textbf{b) } V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\textbf{c) } V = \mathbf{R}^3 \qquad \textbf{d) } V = \{0\}$$

[**Hint:** Only part **a)** requires any work.]

7. Compute the orthogonal decomposition $(3, 1, 3) = b_V + b_{V^\perp}$ with respect to each subspace of V of Problem 1(a)–(e).

[**Hint:** Only parts **a)** and **c)** require any work, and even **c)** doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

8. Let P_V be the matrix for orthogonal projection onto a subspace V of $\mathbf{R}^n$.

- **a)** Explain why $P_V^2 = P_V$.
- **b)** Explain why $P_V + P_{V^\perp} = I_n$.
- **c)** Explain why $P_V^T = P_V$. (Use the formula $P_V = A(A^T A)^{-1} A^T$.)
- **d)** What are $\mathrm{Col}(P_V)$ and $\mathrm{Nul}(P_V)$? What is $\mathrm{rank}(P_V)$? Explain your answers.

9. Compute the matrix P_V for orthogonal projection onto each subspace of Problem 5 and Problem 6. Verify properties (a) and (c) of Problem 8.

10. Compute the matrices P_1, P_2 for orthogonal projection onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$, respectively. Now compute $P_1 P_2$, and explain why it is what it is.

11. Consider the plane V defined by the equation $x + 2y - z = 0$. Compute the matrix P_V for orthogonal projection onto V in two ways:

a) Find a basis for V, put your basis vectors into a matrix A, and use the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $vv^T/v\cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.

[**Hint:** It doesn't take any work to find a basis for $V^\perp$.]

If V is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

12. **a)** Find an implicit equation for the plane

$$\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}.$$

[**Hint:** use Problem 1(a).]

b) Find implicit equations for the line $\{(t,-t,t) : t \in \mathbf{R}\}$.

[**Hint:** use Problem 2(g).]

13. Construct a matrix A with each of the following properties, or explain why no such matrix exists.

a) The column space contains $\begin{pmatrix}0\\2\\1\end{pmatrix}$, and the null space contains $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\2\end{pmatrix}$.

b) The row space contains $\begin{pmatrix}0\\2\\1\end{pmatrix}$, and the null space contains $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\2\end{pmatrix}$.

c) $Ax = \begin{pmatrix}1\\2\\3\end{pmatrix}$ is consistent, and $A^T\begin{pmatrix}-1\\-1\\2\end{pmatrix} = 0$.

d) A 2×2 matrix A with no zero entries such that every row of A is orthogonal to every column.

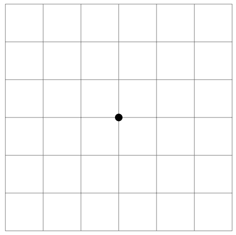
e) The sum of the columns of A is $\begin{pmatrix}0\\0\\0\end{pmatrix}$, and the sum of the rows of A is $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

14. Suppose that S is a symmetric matrix. Explain why $\text{Col}(S)$ is orthogonal to $\text{Nul}(S)$.

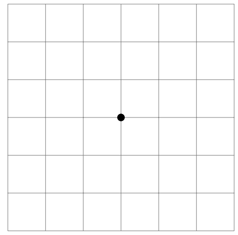
15. Draw the four fundamental subspaces of the following matrices, in grids like below.

a) $\begin{pmatrix}1 & 3\\2 & 6\end{pmatrix}$ **b)** $\begin{pmatrix}1 & 0\\2 & 0\end{pmatrix}$

Row Picture



Column Picture



16. The floor V and the wall W are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet).

a) If $V = \text{Col}(A)$ and $W = \text{Col}(B)$ for

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{pmatrix},$$

find a nonzero vector contained in both V and W.

[**Hint:** You want vectors x and y with $Ax = By$. Form the matrix $(A\ B)$.]

b) Generalize what you did in **a)** to explain why there do not exist orthogonal planes in $\mathbf{R}^3$.

17. Explain why A has full column rank if and only if $A^T A$ is invertible.

18. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal.

b) If A is a 3×4 matrix, then $\text{Col}(A)^\perp$ is a subspace of $\mathbf{R}^4$.

c) If A is any matrix, then $\text{Nul}(A) = \text{Nul}(A^T A)$.

d) If A is any matrix, then $\text{Row}(A) = \text{Row}(A^T A)$.

e) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^\perp$.

f) If x is in V and $V^\perp$, then $x = 0$.

g) If x is in a subspace V, then the orthogonal projection of x onto V is x.