Homework #8

due **Thursday**, October 15, at 11:59pm

1. Compute the determinants of the following matrices *using Gaussian elimination*.

a) $\begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix}$

c) $\begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix}$ **d)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

2. Suppose that

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10 \quad \text{and} \quad \det\begin{pmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{pmatrix} = 5.$$

Find the determinants of the following matrices.

a) $\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$ **b)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix}$ **c)** $\begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix}$

d) $\begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$ **e)** $\begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix}$ **f)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

g) $2\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **h)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **i)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$

j) $-\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **k)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{3}$ **l)** $\begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix}$

m) $\begin{pmatrix} a+2a' & b+2b' & c+2c' \\ d & e & f \\ g & h & i \end{pmatrix}$

3. Find $\det(E)$ when:

a) E is the elementary matrix for a row replacement.

b) E is the elementary matrix for $R_i \times\!= c$.

c) E is the elementary matrix for a row swap.

4. A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

What is $\det(A)$?.

5. Let A be the $n \times n$ matrix $(n \geq 3)$ whose (i, j) entry is $i + j$. Use row operations to show that $\det(A) = 0$.

6.

a) Compute the determinants of the matrices in Problem 1 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!

b) Compute the determinants of the matrices in Problem 1(b) and (d) *again* using Sarrus' scheme.

c) For the matrix of Problem 1(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus' scheme. Did you get the determinant?

7. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

a) Compute the cofactor matrix C of A.

b) Compute AC^T. What is the relationship between C^T and A^{-1}?

8. Consider the $n \times n$ matrix F_n with 1's on the diagonal, 1's in the entries immediately below the diagonal, and -1's in the entries immediately above the diagonal:

$$F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \cdots.$$

a) Show that $\det(F_2) = 2$ and $\det(F_3) = 3$.

b) Expand in cofactors to show that $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$.

c) Compute $\det(F_4), \det(F_5), \det(F_6), \det(F_7)$ using **b)**.

This shows that $\det(F_n)$ is the nth *Fibonacci number*. (The sequence usually starts with $1, 1, 2, 3, \ldots$, so our $\det(F_n)$ is the usual $n + 1$st Fibonacci number.)

9. Let A be an $n \times n$ invertible matrix with integer (whole number) entries.

a) Explain why $\det(A)$ is an integer.

b) If $\det(A) = \pm 1$, show that A^{-1} has integer entries.

c) If A^{-1} has integer entries, show that $\det(A) = \pm 1$.

10. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns.

a) Prove that every orthogonal matrix has determinant ± 1.

b) Prove that the cofactor matrix of an orthogonal matrix Q is $\pm Q$.

c) Show that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal, and compute its determinant.

d) Let L be the line through $(1,1,1)$, and let $R_L = I_3 - 2P_L$ be the reflection over the plane $x + y + z = 0$. Compute R_L, show that it is orthogonal, and find its determinant.

e) Let L be any line in $\mathbf{R}^3$, and let $R_L = I_3 - 2P_L$ be the reflection over the orthogonal plane. Verify that R_L is orthogonal, and prove that $\det(R_L) = -1$, as follows: choose an orthonormal basis $\{u_1, u_2\}$ for $L^\perp$, and let $u_3 = u_1 \times u_2$. Show that the matrix A with columns u_1, u_2, u_3 has determinant 1, and that $R_L A$ has determinant -1.
[**Hint:** Recall that $P_L^2 = P_L = P_L^T$.]

11. Let V be a subspace of $\mathbf{R}^n$ and let P_V be the projection matrix onto V.

a) Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.

b) Find $\det(P_V)$ when $V = \mathbf{R}^n$.

12. Let C be the *hypercube* in $\mathbf{R}^4$ with corners $(\pm 1, \pm 1, \pm 1, \pm 1)$. Compute the volume of C.

13. Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$.

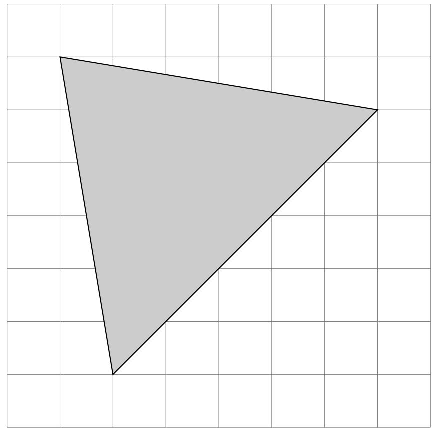
a) Show that if $\{v_1, v_2, \ldots, v_n\}$ is orthogonal then $|\det(A)| = \|v_1\|\,\|v_2\| \cdots \|v_n\|$.
[**Hint:** Compute $A^T A$ and its determinant.]

b) If V is a subspace and x is a vector with orthogonal projection x_V, show that $\|x_V\| \leq \|x\|$, with equality if and only if $x \in V$.

c) Show that $|\det(A)| \leq \|v_1\|\,\|v_2\| \cdots \|v_n\|$, with equality if and only if the set $\{v_1, v_2, \ldots, v_n\}$ is orthogonal.
[**Hint:** Use **b)** and the QR decomposition of A.]

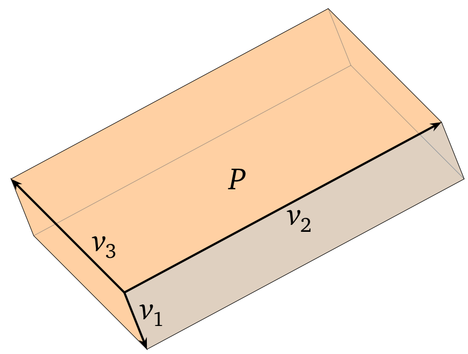
d) What is the largest possible volume of a paralellepiped spanned by four corners of the hypercube C of Problem 12?

14. Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



15. Consider the paralellepiped P in $\mathbf{R}^3$ spanned by

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}-1\\2\\1\end{pmatrix} \qquad v_3 = \begin{pmatrix}0\\-1\\1\end{pmatrix}.$$



a) Compute the volume of P using a triple product $(v_1 \times v_2) \cdot v_3$.

b) Compute the area of each face of P using cross products.

c) If the "base" of P is the paralellogram spanned by v_1 and v_2 (blue in the picture), show that the height of P is $\|v_3\| \sin\theta$, where θ is the angle that v_3 makes with the base. (Draw a simpler picture.)

d) The volume of P is the area of the base of P times its height. How do you reconcile **c)** with **a)**? (Remember that $\|u \cdot v\| = \|u\|\,\|v\| \cos(\text{the angle from } u \text{ to } v)$.)

16. Use a cross product to find an implicit equation for the plane

$$V = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\}.$$

Compare Problem 12(a) in Homework 6.

17. **a)** Let $v = (a,b)$ and $w = (c,d)$ be vectors in the plane, and let $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. By taking the cross product of $(a,b,0)$ and $(c,d,0)$, explain how the right-hand rule determines the sign of $\det(A)$.

b) Using the identity

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

explain how the right-hand rule determines the sign of a 3×3 determinant.

18. Decide if each statement is true or false, and explain why.

a) $\det(A+B) = \det(A) + \det(B)$.

b) $\det(ABC^{-1}) = \dfrac{\det(A)\det(B)}{\det(C)}$.

c) $\det(AB) = \det(BA)$.

d) $\det(3A) = 3\det(A)$.

e) If A^5 is invertible then A is invertible.

f) The determinant of A is the product of its diagonal entries.

g) If the columns of A are linearly dependent, then $\det(A) = 0$.

h) The determinant of the cofactor matrix of A equals the determinant of A.

i) If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.

j) $u \times v = v \times u$.

k) If $u \times v = 0$ then $u \perp v$.