Eg

$$3x + 2y = 2$$
$$x - y = 1$$

Solve for (x, y).

Subtract $3 \cdot$ (2^{nd} equ)
from 1^{st} equ:

$$0 + 5y = -1$$
$$y = -1/5$$

Substitute:

$$x + 1/5 = 1$$

$$x = 4/5$$

$$(x, y) = (4/5, 1/5)$$

- Eliminate variables
- Substitution

Disadvantage: creates a long list of equs.

New method

$3x + 2y = 2$ ①

$x - y = 1$

$3x + 2y = 2$

$3x - 3y = 3$ *

subtract 1st row from second row

$$3x + 2y = 2$$

$$-5y = 1$$

$$3x + 2y = 2$$

$$y = -1/5$$

$$3x = 12/5$$
$$y = -1/5$$

$$x = 4/5$$
$$y = -1/5$$

<u>Idea</u> Replace linear systems by simpler, <u>equivalent</u> linear systems.

have same solutions

The building blocks are <u>elementary row operations</u>.

1) (Row scaling)

Multiply a row by a non-zero number

$$\left\{\begin{array}{c} \vdots \\ 3x + 9y = 6 \\ \vdots \end{array}\right\} \rightsquigarrow \left\{\begin{array}{c} \vdots \\ x + 3y = 2 \\ \vdots \end{array}\right\}$$

2) (Row addition)

Add/subtract a mult. of one row from another.

$$\begin{cases} x + 2y = 3 \\ -2x + y = 0 \end{cases}$$

2R1 + R2 is new R2

$$\Leftrightarrow \begin{cases} x + 2y = 3 \\ 0 + 5y = 6 \end{cases}$$

3) (Row swapping)
Swap 2 eqns

$$\begin{cases} x+y=1 \\ 3x+4y=2 \\ 3y+z=0 \end{cases}$$

Swap 1st and 3rd rows

$$3y+z=0$$
$$3x+4y=2$$
$$x+y=1.$$

why do these ops.

give equivalent linear systems?

<u>Reversible</u> : we can undo them

by another elt. row op.

Augmented matrices

n - variables

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

m eqns $\quad \vdots$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix: $\underset{m \times n}{A} \cdot \underset{n \times 1}{x} = \underset{m \times 1}{b}$

Eg $\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4/5 \\ -1/5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

What happens to matrix / vector

A , b

when we do row ops?

<u>Eg</u> $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$\begin{aligned} y &= 3 \\ x + 2y &= 4 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x + 2y &= 4 \\ y &= 3 \end{aligned}$$

Since we need to do same ops. to A and b, group together:

$(A \mid b)$ <u>augmented matrix</u>

__Eg__

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$(A|b) = \left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right)$$

Can ignore while doing simplifications / row ops.

$$\left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right)$$

Elimination

When are you done w/ elimination?

Eg

$$x + 3y = 3 \qquad \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 1 & 1 & -1 \end{array}\right)$$

$$x + y = -1$$

$$x + 3y = 3 \qquad \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

$$-2y = -4$$

Triangular shape suggests we are done w/ elim.

$y = 2$

$x + 6 = 3 \quad x = -3$

Eg
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$x + y = 2$

$y + z = 3$

$z = 4$

$y = -1$

$x = 3$

Row-echelon form

A matrix is in
row echelon form (REF)

non-zero

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1) all rows which are entirely zero are below all other rows

2) Circle the 1st non-zero entry in each row.
These entries are called <u>pivots</u> / <u>pivot entries</u>.
The pivots need to be going down and to the right

Eg
$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 4 & 2 \end{array}\right)$$
REF ✓

Eg

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & -5 \end{pmatrix}$$

not in REF,

2^{nd} circled entry needs to be right of 1^{st} circled entry

Eg

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

REF ✓

Eg

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & 5 & 1 \end{array}\right)$$

$$2x+3y=7$$
$$5y=1$$

exactly 1 solution

no solution

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & 0 & 1 \end{array}\right)$$

$$2x+3y=7$$
$$0=1$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 0 & 0 & 5 & 1 \end{array}\right)$$

$$2x+3y+z=7$$
$$5z=1$$

no pivot in col

∞ly many solutions

1 solution $\Longleftrightarrow$ triangular form, all pivots are in the A part of aug. matrix and every column in A-part has a pivot

∞ly solution $\Longleftrightarrow$ all pivots are in A part but col. has no pivots

0 solutions $\longleftrightarrow$ a pivot is in the b-part.

Eg
$$\begin{aligned} x+y&=1 \\ x+y&=2 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|c}1&1&1\\1&1&2\end{array}\right) \sim \left(\begin{array}{cc|c}1&1&1\\0&0&1\end{array}\right)$$

$$\begin{aligned} x+y&=1\\ 0&=1 \end{aligned}$$