

Eg

$$\begin{cases} 3x + 2y = 2 \\ x - y = 1 \end{cases}$$

Solve for (x, y) .

Subtract $3 \times$ (2nd equ)
from 1st equ:

$$0 + 5y = -1$$

$$y = -1/5$$

Substitute:

$$x + \frac{1}{5} = 1$$

$$x = \frac{4}{5}$$

$$(x, y) = \left(\frac{4}{5}, \frac{1}{5}\right)$$

- Eliminate variables
- Substitution

Disadvantage: creates a long list of eqns.

New method

$$3x + 2y = 2$$

$$x - y = 1$$



$$3x + 2y = 2$$

$$3x - 3y = 3$$



↓ Subtract 1st row
From second
row

$$3x + 2y = 2$$

$$-5y = 1$$

↓

$$3x + 2y = 2$$

$$y = -\frac{1}{5}$$



$$3x = 12/5$$

$$y = -1/5$$



$$x = 4/5$$

$$y = -1/5$$

Idea

Replace linear systems

by simpler, equivalent

linear systems.



The building blocks all have same solutions
elementary row operations.



1) (Row scaling)

Multiply a row by

a non-zero number

$$\left\{ \begin{array}{c} \vdots \\ 3x + 9y = 6 \\ \vdots \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \vdots \\ x + 3y = 2 \\ \vdots \end{array} \right.$$

2) (Row addition)

Add/subtract a mult. of

one row from another.

$$\begin{cases} x + 2y = 3 \\ -2x + y = 0 \end{cases}$$

\downarrow $2R1 + R2$ is
new $R2$

$$\begin{cases} x + 2y = 3 \\ 0 + 5y = 6 \end{cases}$$

3) (Row swapping)
Swap 2 eqns

$$\begin{cases} x + y = 1 \\ 3x + 4y = 2 \\ 3y + z = 0 \end{cases}$$

Swap 1st and 3rd rows

$$3y + z = 0$$

$$3x + 4y = 2$$

$$x + y = 1.$$

why do these ops.

give equivalent linear
systems?

Reversible : we can undo them
by another eH. law op.

Augmented matrices

a -variables

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

\vdots

m eqns

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Matrix

$$\underbrace{\begin{bmatrix} A & \cdot & x & = & b \end{bmatrix}}_{\substack{m \times n & & n \times 1 & & m \times 1}}$$

Eg $\left(\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -1 & -1/5 \end{array} \right) = \left(\begin{array}{c} 2 \\ 1 \end{array} \right)$

What happens to matrix/vector
A, b
when we do row ops?

Eg $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$\begin{array}{l} y = 3 \\ x + 2y = 4 \end{array} \sim \begin{array}{l} x + 2y = 4 \\ y = 3 \end{array}$$

Since we need to do same ops.
to A and b , group together:

$(A|b)$ augmented matrix

Eg

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$(A|b) = \left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array} \right)$$

Can ignore \rightarrow while doing
simplifications / row ops.

$$\left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array} \right)$$

Elimination

when are you done

w/ elimination?

Eg $x + 3y = 3$

$$x + y = -1$$



$$\begin{array}{r} x + 3y = 3 \\ x + 2y = -4 \end{array}$$



$$\left(\begin{array}{cc|c} 1 & 3 & 3 \\ 1 & 1 & -1 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -2 & -4 \end{array} \right)$$

↑ triangular shape suggests we are done w/ elim.

$$y = 2$$

$$x + 6 = 3$$

$$x = -3$$

Eg

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

$$x + y = 2$$

$$y + z = 3$$

$$z = 4$$

$$y = -1$$

$$x = 3$$

Row - echelon form

A matrix is in

row echelon form (REF)

non-zero

$$\left(\begin{array}{cccc} \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 \end{array} \right)$$

- 1) all rows which are entirely zero are below all other rows

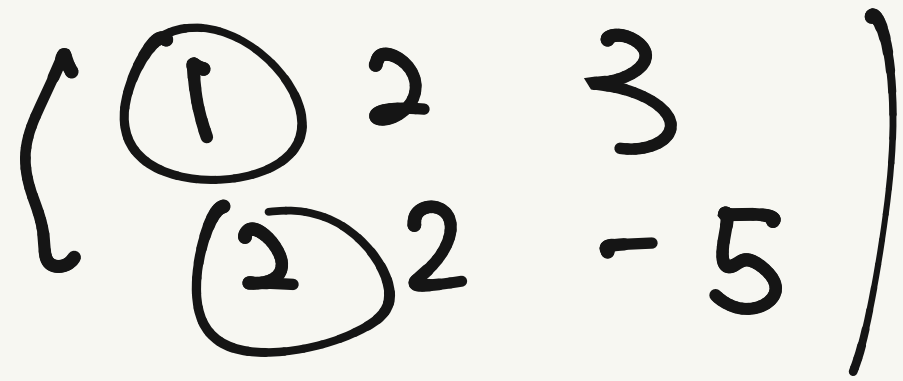
2) Circle the 1st non-zero entry in each row.

These entries are called pivots / pivot entries.

The pivots need to be going down and to the right

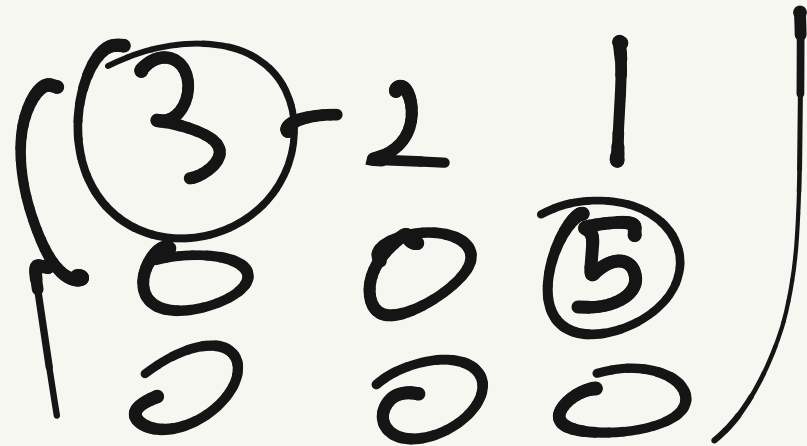
Eg $\left(\begin{array}{cc|c} \textcircled{1} & 2 & 1 \\ 0 & \textcircled{4} & 2 \end{array} \right)$ P E F \checkmark

Eg



not in REF,
2nd circled entry needs to
be right of 1st circled entry

Eg



REF ✓

Eg $\rightarrow \left(\begin{array}{cc|c} 2 & 3 & 7 \\ 2 & 5 & 1 \end{array} \right)$

$$2x + 3y = 7$$

$$5y = 1$$

exactly 1 solution

no solution $\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & 0 & 1 \end{array} \right)$

$$2x + 3y = 7$$

$$0 = 1$$

$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 7 \\ 0 & 0 & 5 & 1 \end{array} \right)$

$$2x + 3y + z = 7$$

$$5z = 1$$

no pivot \rightarrow 2y many solutions
in col

| solution \Leftrightarrow triangular form,
all pivots are in
the A part of
aug. matrix
and every column
in A-part has
a pivot

∞ly solution \Leftrightarrow all pivots are
in A part
but col. has
no pivots

○ solutions \Leftrightarrow a pivot is in the b-part.

$$\begin{array}{l} \text{Eg} \\ x + y = 1 \\ x + y = 2 \end{array} \quad \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array} \right)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

2

$$\begin{cases} x + y = 1 \\ z = 1 \end{cases}$$