Row echelon form:

1) zero rows at bottom
2) first non-zero entry in each row is <u>pivot</u>
3) pivots must go down and to the right
4) all entries <u>below</u> a pivot entry are zero.

Gaussian elimination compute REF

we go one column at a time, starting from the left.

Look in the current col. for the first non-zero entry in a row which doesn't have a pivot.

Circle that entry - it will be next pivot.

Row swap to move this circled entry into the

1st row which didn't already have a pivot.

Row additions to make all entries below our current pivot zero.

Move on to the next column.

Eg $\begin{pmatrix} 1 & 4 & 3 \\ 0 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$ $R2 \leftrightarrow R3$ $\begin{pmatrix} 1 & 4 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Eg $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$ $R3 -= 4 \cdot R2$ $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{pmatrix}$

Eg $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Row reduced echelon form

1) REF
2) Pivot entries are all equal to 1.

3) All entries above pivot must be zero.

Eg $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ REF
not RREF
2 above 1 is not zero.

Eg $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I_2}$ RREF

Eg $\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$ RREF

Eg $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ REF, not RREF
2^{nd} pivot $\neq 1$

Point of RREF:

a matrix in RREF is simplified as much as possible.

Each matrix has a unique RREF.

Eg
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right) \leftrightarrow \begin{aligned} x_1 &= 4 \\ x_2 &= 5 \\ x_3 &= 6 \end{aligned}$$

To convert any matrix to RREF:

1) Gauss. elim. to obtain REF

2) Use row scaling to make all pivots $= 1$

3) Jordan substitution:

use row addition to make entries above pivots $= 0$.

Look at rows, from bottom to the top.

Use the current row in row addition to make all entries above pivot zero.

$$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R2 -= 4 \cdot R3 \quad R1 -= 6 \cdot R3$$

$$\begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R1 -= 2R2$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad RREF$$

Gauss - Jordan algorithm

Doing row ops. using matrix mult.

Eg $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$

Eg $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+3c & b+3d \\ c & d \end{pmatrix}$

$R1 += 3R3$

Eg $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

Elt matrix = matrix which performs an elt. row op.

How to read what a matrix does:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

New R1 = old R1 − 2·old R3

New R2 = old R2

New R3 = old R3

Eg $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ new R1 = old R2

new R2 = old R1

Elementary row operations are reversible

Eg new R1 → new R2 → $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$

Reverse our op.

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

mult. these

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the inverse of

Invertible matrices

A square matrix A ($n \times n$) is invertible if there exists an $n \times n$ matrix B so that

$$BA = I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Invertible matrices $\leftrightsquigarrow$ reversible operations

Fact 1) If A is invertible, there is only one matrix B such that $BA = I_n$.

$\parallel$

A^{-1} is the *inverse*

$A^{-1} \cdot A = I_n$

2) $\underline{A \cdot A^{-1} = I_n}$ also.

3) $(A^{-1})^{-1} = A$

Eg $\underline{1\times 1}$

$$(a)^{-1} = (a^{-1}) = \left(1/a\right)$$

1×1 matrix is invertible if and only if $a \neq 0$.

$\underline{2\times 2}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Memorize this!

is invertible if and only if $ad-bc \neq 0$.

Eg $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

<u>How to compute the inverse:</u>

A $n \times n$

$(A \mid I_n)$ $n \times 2n$

Compute RREF of this.

If A is invertible, you get:

rref of $\underbrace{(A \mid I_n)}$ is $(I_n \mid A^{-1})$.