A square $n \times n$ matrix A is **invertible** if it has an inverse A^{-1} satisfies:

$$A^{-1}A = I_n$$

or

$$A A^{-1} = I_n .$$

Properties

1) If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.

2) If A and B are invertible then AB is invertible

$\boxed{AB \neq BA}$ usually

and $(AB)^{-1} = \underline{B^{-1}A^{-1}}$

3) A is invertible if and only if A has n pivots

i.e. $\text{rref}(A) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$

(the only $n \times n$ RREF w/ n pivots)

Check 2)

$$(B^{-1}A^{-1})\cdot(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}\underbrace{I_n B}$$

For any matrix $I_m C_{m\times n} = C$

$$= B^{-1}B = I_n$$

$$B^{-1}A^{-1} = (AB)^{-1}.$$

Property 3

Start w/ any $n \times n$ A.

G-J: do row ops to A to convert to $\mathrm{rref}(A)$,

i.e. find elt. matrices

$$\underbrace{(E_k \cdots E_2 E_1)}_{E} \cdot A = \mathrm{rref}(A)$$

Claim E is invertible

Why? (Each E_i is invertible.)

$\therefore$ $E \cdot A = \text{rref}(A)$ and E is invertible

Claim A is invertible precisely when its $\text{rref}(A)$ is invertible

why? • If A is inv. then

$\underset{\text{inv}}{E} \cdot \underset{\text{inv}}{A}$ is, $EA = \text{rref}(A)$

- If $\underline{\text{rref}}(A)$ is inv.

$$EA = \text{rref}(A)$$

$$A = \underbrace{E^{-1}}_{\text{inv}}\underbrace{\text{rref}(A)}_{\text{inv}}$$

Hence A is invertible.

If A is invertible, $\text{rref}(A) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

only RREF matrix w/ n pivots

<u>why</u>: If A is inv.
$\text{rref}(A)$ is inv.

If $\text{rref}(A)$ had $< n$ pivots
it would have a zero row.

Claim Any matrix w/ a zero row is not invertible.

$\therefore$ rref(A) has exactly n pivots.

Two consequences

- If A is invertible, $\text{rref}(A) = I_n$

- If A is invertible the product $E_k \cdots E_1$ of elt matrices

From G-J algorithm equals A^{-1}.

Eg (computing inverses using G-J)

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad E_1 A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad E_2 E_1 A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} \qquad \underbrace{E_3E_2E_1}_{E} A = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\text{rref}(A) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \Rightarrow A \text{ is invertible}$$

$$E = E_3E_2E_1 \text{ is } A^{-1}$$

$$E_3E_2E_1 = \begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}}$$

$$= \begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1/2 & 1/2\end{pmatrix}$$

$$= \begin{pmatrix}2 & -1\\ -1/2 & 1/2\end{pmatrix}$$

An easier method:

Compute RREF of

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right) = \underline{(A \mid I_2)}$$

$$\wr$$

$$\left(\begin{array}{cc|c} 1 & 2 & \overbrace{\begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix}}^{E_1} \\ 0 & 2 & \end{array}\right)$$

$$\wr$$

$$\left(\begin{array}{cc|c} 1 & 2 & \overbrace{\begin{matrix} 1 & 0 \\ -1/2 & 1/2 \end{matrix}}^{E_2 E_1} \\ 0 & 1 & \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & \overbrace{\begin{matrix} 2 & -1 \\ -1/2 & 1/2 \end{matrix}}^{E_3 E_2 E_1} \\ 0 & 1 & \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$$

$$(I_2 \mid A^{-1})$$

Solving nonsingular systems

n variables, n equations

$$\underbrace{A}_{n\times n} x = b$$

Q: when does this have a unique solution?

system is nonsingular

A: Precisely when A is invertible.

$$\underbrace{A^{-1}A}_{I_n} x = A^{-1}b \qquad \boxed{x = A^{-1}b}$$

Condition doesn't depend on b at all.

$\Rightarrow$ $Ax = b$ has a unique sol'n if and only if $Ax = 0$ has a unique sol'n.

A bad algorithm:

1) Compute A^{-1} using GJ

2) Compute $\boxed{x = A^{-1} \cdot b}$.

Computing A^{-1} via GJ:

basically solving

$$Ax = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

then solving

$$Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$\vdots$ etc.

$$(A \mid I_n)$$

n cols in augmented

How do computers solve $Ax=b$?

LU decomposition

$A = LU$ Factorization

Suppose A is invertible and also needs no row swaps when doing G-J.

$$\overbrace{\underbrace{(E_\ell \cdots E_{k+1})}_{\text{subst.}}}^{E_u} \cdot \overbrace{\underbrace{(E_k \cdots E_{j+1})}_{\text{scaling}}}^{E_d} \overbrace{\underbrace{(E_j \cdots E_1)}_{\text{elimination}}}^{E_\ell} \cdot A = I_n$$

- Elimination matrices:
  - lower-triangular

$$n\begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}$$
$$n$$

- Scaling matrices diagonal: $\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

- substitution : upper triangular

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

upper diag. lower

Eg $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In general, take product of groups of matrices

$$E_u \, E_d \, E_l \, A = I_n$$

upper diag lower

$$A = E_\ell^{-1} \underbrace{E_d^{-1} E_u^{-1}}$$

$$= \underset{\text{lower}}{L} \cdot \underset{\text{upper}}{U}$$

$$\boxed{\text{Every invertible } n \times n \text{ matrix which doesn't need row swaps can be factored } A = \underset{\text{lower}}{L}\,\underset{\text{upper}}{U}.}$$

$$\underline{\text{Eg}} \quad A = \overset{E_1^{-1}}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \underbrace{\overset{E_2^{-1}}{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \overset{E_3^{-1}}{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}}_{U}$$

Algorithm for $Ax = b$

1) Find $A = LU$ factorization

$$\left.\begin{array}{l} \text{is as easy as substn} \end{array}\right\} \begin{array}{l} \text{2) Solve } Lc = b \qquad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ \text{3) Solve } U\underline{x} = c \end{array}$$

Eg $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = b$

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}}_{U}$$

$\boxed{Lc = b} \rightsquigarrow \boxed{\begin{aligned} c_1 &= 1 \\ c_1 + c_2 &= -1 \end{aligned}}$

$c_2 = -2$

$$c = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boxed{Ux = c}$$

$$x_1 + 2x_2 = 1 \qquad x_1 = 3$$

$$2x_2 = -2 \qquad x_2 = -1$$

$$\boxed{x = \begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$

I haven't told you a ~~good~~ algorithm for $A = LU$.

$\underline{PA = LU}$

what happens if you need row swaps?

Any invertible A can be factored

$$PA = LU$$

permutation lower upper

A <u>permutation</u> matrix P is a matrix w/ same rows as $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ but in a diff order.

<u>Eg</u> $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$

Discussion after class:

Why is elimination lower-triangular? (when no row swaps)

$$\begin{pmatrix} * & & & 0 \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{pmatrix} \cdot A$$

modify later rows w/ prev. rows.

Substitution:

modify prev. rows w/ later rows

$$\begin{pmatrix} * & * & * & * \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}$$