Complex Numbers

Can you diagonalize $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (ccw rotation by $90°$)?

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(\lambda) = \lambda^2 + 1$$

has no real roots! So there are no real eigenvalues...

Def: The unit imaginary number is a number i such that $i^2 = -1$. A complex number is a number $a+bi$ for $a, b \in \mathbb{R}$.

$$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$$

Real part: $\mathrm{Re}(a+bi) = a$

Imaginary part: $\mathrm{Im}(a+bi) = b$

Algebra:

- $(a+bi) + (c+di) = (a+c) + (b+d)i$
- $(a+bi)(c+di) = ac + bdi^2 + adi + bci$
  $= (ac - bd) + (ad + bc)i$

Conjugation: $\overline{a+bi} = a - bi$ ie. $i \longmapsto -i$

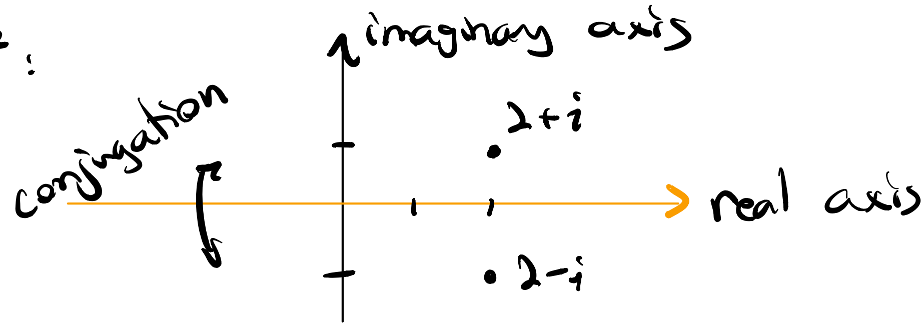
- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$

$$(a+bi) + \overline{(a+bi)} = a+bi+a-bi = 2a = 2\mathrm{Re}(a+bi)$$
$$(a+bi) - \overline{(a+bi)} = a+bi-a+bi = 2bi = 2i\,\mathrm{Im}(a+bi)$$

$$\Longrightarrow \boxed{\begin{aligned} Re(z) &= \tfrac{1}{2}(z+\bar z) \\ Im(z) &= \tfrac{1}{2i}(z-\bar z) \end{aligned}}$$

Draw $\mathbb{C}$ as $\mathbb{R}^2$:

complex plane

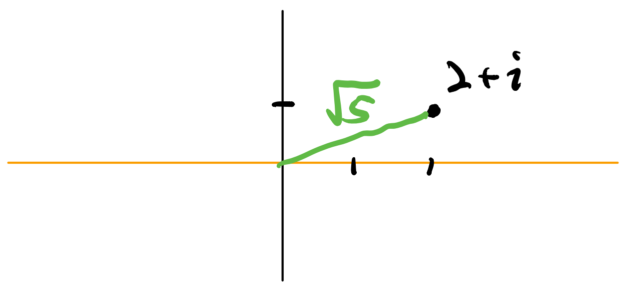


NB: $\mathbb{C}$ contains a copy of $\mathbb{R}$: all $a+0i$

NB: $(a+bi)\overline{(a+bi)} = (a+bi)(a-bi) = a^2-(ib)^2$
$= a^2 - i^2b^2 = a^2+b^2 \geq 0$

Def: The modulus of z is $|z| = \sqrt{z\bar z}$ "length of z"

Eg: $|2+i| = \sqrt{4+1}$
$= \sqrt{5}$



Eg: $|a| = \sqrt{a^2+0^2} = |a|$

(real number; usual absolute value)

NB: $|zw|^2 = (zw)\overline{(zw)} = zw\bar z\bar w = z\bar z w\bar w = |z|^2|w|^2$
$\Longrightarrow |zw| = |z|\cdot|w|$

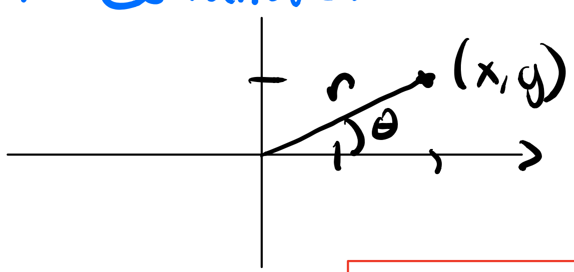
Division: $\boxed{\dfrac{1}{z} = \dfrac{1}{z}\cdot\dfrac{\bar z}{\bar z} = \dfrac{\bar z}{|z|^2}}$ ← real number

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

Eg: $$\frac{1}{2+i} = \frac{2-i}{2^2+1^2} = \frac{2}{5} - \frac{1}{5} i$$

Check: $(2+i)\left(\frac{2}{5} - \frac{1}{5} i\right) = \frac{4}{5} + \frac{1}{5} + \left(\frac{2}{5} - \frac{2}{5}\right) i = 1$ ✓

Polar Coordinates



$$r = \sqrt{x^2+y^2}$$

$$\theta = \pm \arctan\left(\frac{y}{x}\right)$$

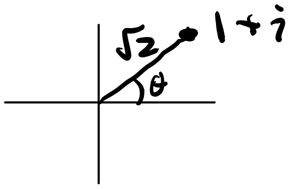
$$\boxed{(r, \theta) \rightsquigarrow \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

Complex numbers:

$$z = |z| (\cos\theta + i \sin\theta)$$

θ = argument of z = $\arg(z)$

Eg:



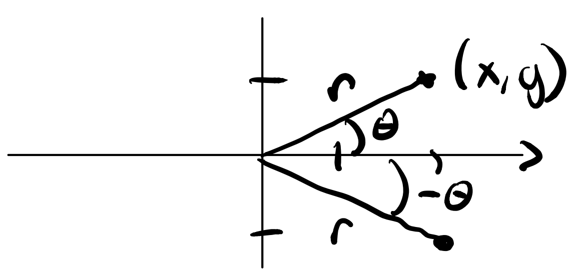
$$1+i = \sqrt{2}(\cos 45° + i \sin 45°)$$

$$\arg(1+i) = 45°$$

NB: $\arg(\bar{z}) = -\arg(z)$

Since $\frac{1}{z} = \frac{1}{|z|^2} \bar{z}$,

$$\arg\left(\frac{1}{z}\right) = -\arg(z)$$



Fact: $\arg(zw) = \arg(z) + \arg(w)$ $\qquad |zw| = |z|\cdot|w|$

This is a trig identity!

$$z = |z|(\cos\theta + i\sin\theta)$$
$$w = |w|(\cos\varphi + i\sin\varphi)$$
$$zw = |z|\cdot|w|(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$
$$= |zw|(\cos\theta\cos\varphi - \sin\theta\sin\varphi + (\cos\theta\sin\varphi + \sin\theta\cos\theta)i)$$

sum angle formulas

$$= |zw|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$$
$$\Rightarrow \theta + \varphi = \arg(zw) \qquad \theta = \arg(z) \quad \varphi = \arg(w)$$

So multiplication is convenient in polar form!

$$(|z|, \arg(z))\cdot(|w|, \arg(w)) = (|z|\cdot|w|, \arg(z) + \arg(w))$$

Euler's Formula: for $\theta \in \mathbb{R}$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$\rightarrow$ take Taylor expansions of both sides...

Polar Form, Alternate: $\qquad z = |z|\cdot e^{i\arg(z)}$

$$zw = |z|e^{i\arg(z)}|w|e^{i\arg(w)}$$
$$= |zw|e^{i(\arg(z)+\arg(w))}$$
$$\frac{1}{z} = \frac{1}{|z|e^{i\arg(z)}} = \frac{1}{|z|}e^{-i\arg(z)}$$

NB: $\overline{e^{i\theta}} = e^{-i\theta}$

$$\overline{z} = \overline{|z|e^{i\arg(z)}} = |z|\,e^{-i\arg(z)}$$

Fundamental Theorem of Algebra:

Every polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n, a_{n-1}, \ldots, a_n \in \mathbb{C}$ $a_n \neq 0$ factors as

$$f(x) = a_n(x-\lambda_1)\cdots(x-\lambda_n) \quad \text{for some } \lambda_1, \ldots, \lambda_n \in \mathbb{C}$$

Eg: $f(x) = x^2 + x + 1$

$$\sqrt{-1}\cdot\sqrt{3}$$

$$\Rightarrow x = \frac{1}{2}\left(-1 \pm \sqrt{1-4}\right) = \frac{1}{2}\left(-1 \pm \sqrt{-3}\right) = \frac{1}{2}(-1 \pm 3i)$$

$$\Rightarrow f(x) = \left(x - \tfrac{1}{2}(-1+3i)\right)\left(x - \tfrac{1}{2}(-1-3i)\right)$$

Fact: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n, a_{n-1}, \ldots, a_0 \in \mathbb{R}$ (has real coefficients) then

$$f(\overline{z}) = a_n \overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1\overline{z} + a_0$$
$$= \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$
$$= \overline{f(z)}$$

In particular, $f(z) = 0 \Leftrightarrow f(\overline{z}) = 0$

Complex roots of real polynomials come in conjugate pairs.