Geometry of Diagonalizable Matrices

So far:

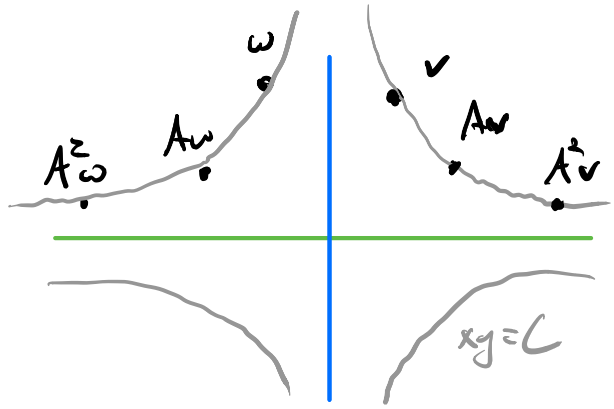
- $A^m(x_1w_1+\cdots+x_nw_n) = \lambda_1^m x_1 w_1 + \cdots + \lambda_n^m x_n w_n$
- $A = CDC^{-1} \rightsquigarrow A^m = CD^mC^{-1}$

What do these say geometrically?

Eg: $D=\begin{pmatrix}2&0\\0&1/2\end{pmatrix}$ $\quad D\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x\\ 1/2\, y\end{pmatrix}$

Note: $xy = (2x)(1/2\, y)$

$\Rightarrow \begin{pmatrix}x\\y\end{pmatrix}$ & $D\begin{pmatrix}x\\y\end{pmatrix}$ lie on the hyperbola $xy=C$



Eg: $A=\frac{1}{10}\begin{pmatrix}11&6\\9&14\end{pmatrix}$ $\quad p(\lambda)=\lambda^2-\frac52\lambda+1=(\lambda-2)(\lambda-1/2)$

$\lambda_1=2 \quad w_1=\begin{pmatrix}2\\3\end{pmatrix} \qquad \lambda_2=1/2 \quad w_2=\begin{pmatrix}-1\\1\end{pmatrix}$

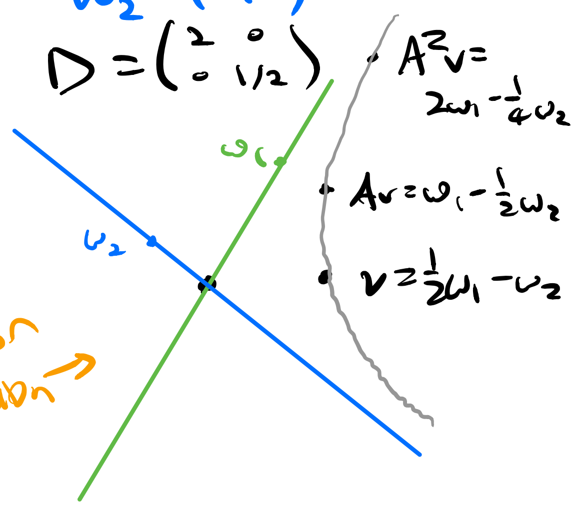
$\Rightarrow A=CDC^{-1} \quad C=\begin{pmatrix}2&-1\\3&1\end{pmatrix} \quad D=\begin{pmatrix}2&0\\0&1/2\end{pmatrix}$

What does this do?

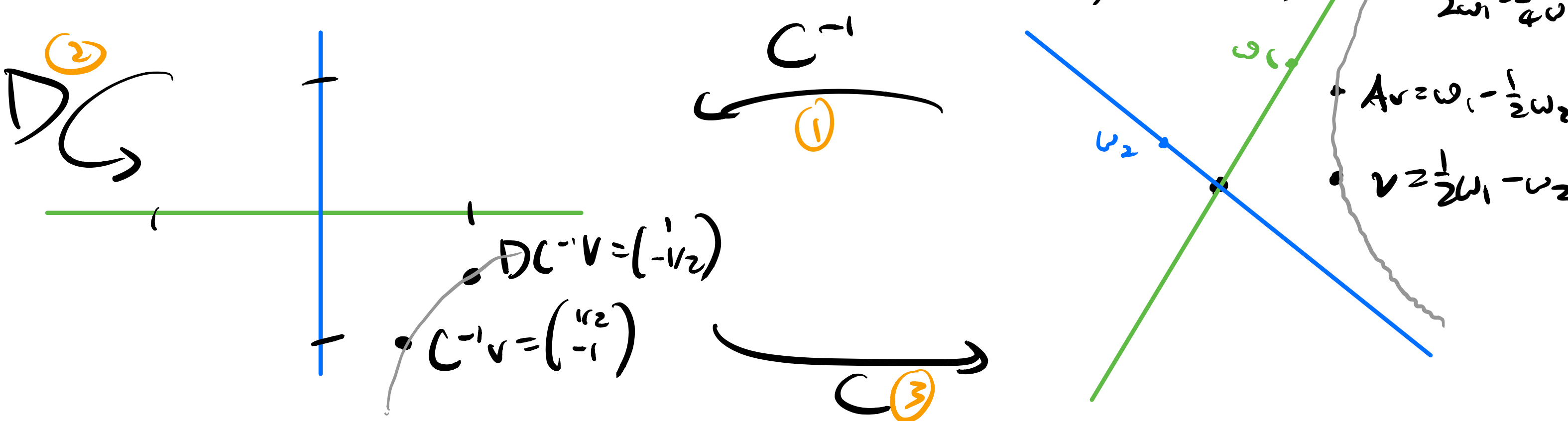
Work in the eigenbasis!

$\rightarrow$ Think in terms of LC's of w_1, w_2

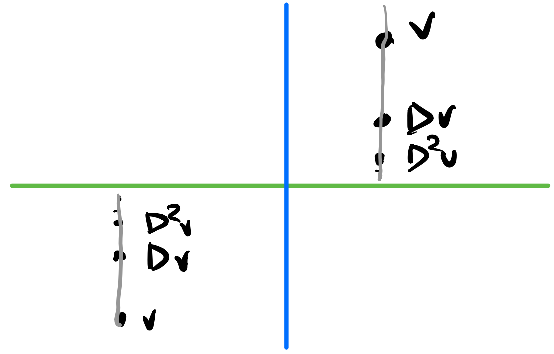
vector version $\rightarrow$



Matrix version: $A=CDC^{-1}$ $\quad C^{-1}(x_1w_1+x_2w_2)=\begin{pmatrix}x_1\\x_2\end{pmatrix}$

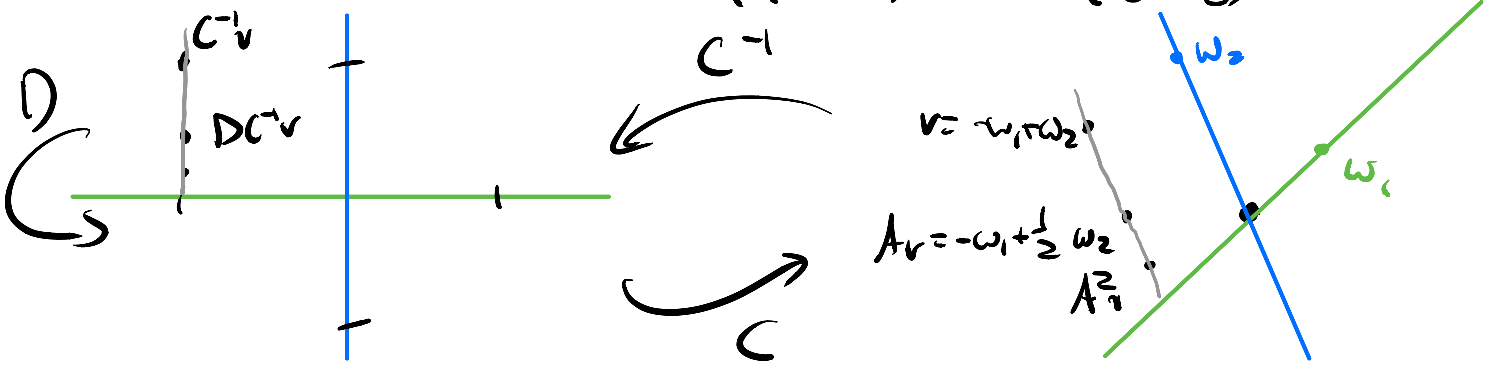


Eg: $D=\begin{pmatrix}1&0\\0&1/2\end{pmatrix}$ $\quad D\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\1/2\,y\end{pmatrix}$



Eg: $A=\frac{1}{6}\begin{pmatrix}5&1\\2&4\end{pmatrix}$ $\quad \lambda_1=1 \quad w_1=\begin{pmatrix}1\\1\end{pmatrix} \quad \lambda_2=1/2 \quad w_2=\begin{pmatrix}-1\\2\end{pmatrix}$

$A=CDC^{-1} \qquad C=\begin{pmatrix}1&-1\\1&2\end{pmatrix} \qquad D=\begin{pmatrix}1&0\\0&1/2\end{pmatrix}$



(almost) picture of a stochastic matrix

→ look at other demos

Eg: $A=\frac{1}{580}\begin{bmatrix}503&73&269\\207&1137&-49\\270&-30&680\end{bmatrix}=CDC^{-1}$

$D=\begin{bmatrix}1/2&0&0\\0&2&0\\0&0&3/2\end{bmatrix} \qquad C=\frac{1}{6}\begin{bmatrix}-7&-1&2\\2&-9&-1\\5&0&3\end{bmatrix}$

Diagonalization with Complex Numbers

Diagonalization works great even if eigenvalues aren't real.

→ Still can solve difference equations / ODE's!

Fact: If A is a real matrix & $Av=\lambda v$ $\lambda\in\mathbb{C} \Rightarrow A\bar{v}=\bar{\lambda}\bar{v}$

Complex eigenvals/eigenvecs come in conjugate pairs!

Eg: Solve the difference equation

$$v_{n+1} = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} v_n \qquad v_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

← no complex numbers

Solve by diagonalization:

$$A = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \qquad p(\lambda) = \lambda^2 + 3\lambda + 3 \qquad \lambda = \frac{-3 \pm \sqrt{9-12}}{2}$$

$$\lambda = \tfrac{1}{2}(-3 + i\sqrt{3}) \qquad \bar{\lambda} = \tfrac{1}{2}(-3 - i\sqrt{3})$$

$$A - \lambda I_2 = \begin{pmatrix} \frac{1}{2}(3 - i\sqrt{3}) & -1 \\ \_\_\_ & \_\_\_ \end{pmatrix} \xrightarrow{\text{HW\#9.2a}} w = \begin{pmatrix} 1 \\ \frac{1}{2}(3 - i\sqrt{3}) \end{pmatrix}$$

Eigenvector for $\bar{\lambda}$ $\longrightarrow$ $\bar{w} = \begin{pmatrix} 1 \\ \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix}$

$$\Rightarrow A = CDC^{-1} \quad C = \begin{pmatrix} 1 & 1 \\ \frac{1}{2}(3 - i\sqrt{3}) & \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix} \quad D = \begin{pmatrix} \frac{1}{2}(-3 + i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(-3 - i\sqrt{3}) \end{pmatrix}$$

To compute $v_n = A^n v_0$, need $v_0 = x_1 w_1 + x_2 w_2 \rightsquigarrow x_1 = x_2 = 1$

$$v_n = A^n v_0 = A^n(w + \bar{w}) = \lambda^n w + \bar{\lambda}^n \bar{w}$$

$$v_n = \left(\frac{-3 + i\sqrt{3}}{2}\right)^n \begin{pmatrix} 1 \\ \frac{1}{2}(3 - i\sqrt{3}) \end{pmatrix} + \left(\frac{-3 - i\sqrt{3}}{2}\right)^n \begin{pmatrix} 1 \\ \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix}$$

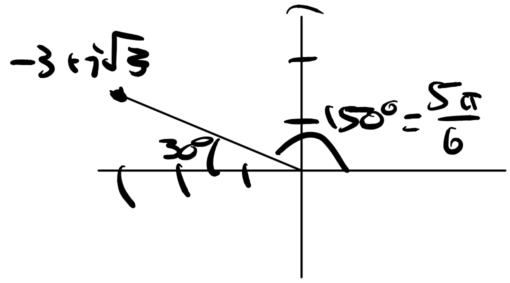
→ So far, exactly the same as real eigenvalues.

How to get rid of imaginary numbers in this answer?

Say $v_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$. Let's compute x_n.

$$x_n = \frac{1}{2^n}\Big[\underbrace{(-3 + i\sqrt{3})^n + (-3 - i\sqrt{3})^n}_{\text{conjugates}}\Big] = \frac{1}{2^n} \cdot 2\,\mathrm{Re}\left[(-3 + i\sqrt{3})^n\right]$$

Trick: use polar coordinates!



$$-3+i\sqrt{3} = re^{i\theta}$$

$$r = |-3+i\sqrt{3}| = \sqrt{12} = 2\sqrt{3}$$

$$\theta = 5\pi/6$$

$$x_n = \frac{1}{2^n}\cdot 2\,\mathrm{Re}\left[(2\sqrt{3})e^{5\pi i/6}\right]^n = \frac{1}{2^n}\cdot 2\,\mathrm{Re}\left[(2\sqrt{3})^n e^{5\pi i n/6}\right]$$

$$= \frac{1}{2^n}\cdot 2\cdot(2\sqrt{3})^n \cos\left(\frac{5\pi n}{6}\right)$$

$$x_n = 2\cdot(\sqrt{3})^n\cdot \cos\left(\frac{5\pi n}{6}\right)$$

Complex Diagonalization (of a real matrix)

- Do exactly the same as real diagonalization
- Remember $Av = \lambda v \iff A\bar{v} = \bar{\lambda}\bar{v}$
- To get only real numbers in your answer, use polar coordinates & Euler's formula to compute
  $$\mathrm{Re}(\lambda^n \cdot x_1 w_1 + \cdots)$$