Algebraic & Geometric Multiplicity

Want a criterion for diagonalizability.

Def: Let A be an $n\times n$ matrix with eigenvalue λ.

(1) The algebraic multiplicity of λ is its multiplicity as a root of $p(\lambda)$: the power of $(x-\lambda)$ dividing $p(x)$.

(2) The geometric multiplicity of λ is the dimension of the λ-eigenspace: $\dim \mathrm{Nul}(A-\lambda I)$.

Fact (AM≥GM):

there is an eigenvector

algebraic multiplicity of $\lambda \geq$ geometric multiplicity $\boxed{\geq 1}$

Eg: $A=\begin{pmatrix}-7 & 3 & 5\\ -10 & 5 & 6\\ -9 & 3 & 7\end{pmatrix}$ $\quad p(\lambda)=-(\lambda-2)^{2}(\lambda-1)^{1}$

$\lambda=1$: $AM=1 \Longrightarrow GM=1$ i.e. 1-eigenspace is a line!

$1=AM\geq GM\geq 1$

$\lambda=2$: $AM=2 \Longrightarrow GM=1$ or 2

Compute $\mathrm{Nul}(A-2I_3)=\mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}\right\} \Longrightarrow GM=1$

AM/GM Criterion for Diagonalizability: Let A be an $n\times n$ matrix. Then A is diagonalizable (over $\mathbb{C}$) $\Longleftrightarrow$ AM=GM of each eigenvalue.

Diagonalizable over $\mathbb{R}$: also need all real eigenvalues.

Why? Sum all AM's $= \deg p(\lambda) = n$.

AM=GM $\iff$ sum of GM's $= n$

$\iff$ have n LI eigenvectors

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ for $\lambda=2$: AM $=2 > 1 =$ GM

$\Rightarrow$ not diagonalizable.

Corollary: If A has n different eigenvalues then A is diagonalizable.

$\rightarrow$ means $p(\lambda)$ has all simple roots $\Rightarrow$ AM=GM=1.

Eg: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$ is diagonalizable:

eigenvalues are $1, 5, 8, 10$

Systems of Ordinary Differential Equations (ODEs).

- $\rightarrow$ Highly analogous to difference equations ("continuous time")
- $\rightarrow$ Important application of diagonalization
- $\rightarrow$ Today: just a taste.

Simplest case: solve for a function $u(t)$ s.t.

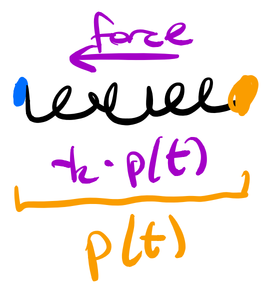
$u' = \lambda u \qquad u(0) = u_0 \in \mathbb{R}$

Eg: Continuously compounded interest: money in your savings account increases u' continuously in proportion to the amount $u(t)$ in it. $\lambda \leftrightarrow$ interest rate.

Solution: $u(t) = u_0 \cdot e^{\lambda t}$ check: $u(0) = u_0$ $u' = \lambda \cdot u_0 e^{\lambda t}$
$= \lambda u$ ✓

Hooke's Law: $p''(t) = -k \cdot p(t)$ $k > 0$

Governs physics of springs:



Specify initial position & velocity:

$$p(0) = p_0 \qquad p'(0) = v_0$$

Create a system of ODEs: $v = p' =$ velocity

$$\begin{aligned} p' &= v \\ v' &= -kp \end{aligned} \qquad \begin{bmatrix} p' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} \qquad \begin{bmatrix} p(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} p_0 \\ v_0 \end{bmatrix}$$

Def: A system of ODEs is

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} \quad \text{unknown functions} \quad \text{s.t. } u' = \begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix} = Au$$

for an $n \times n$ matrix A. An initial condition means specifying $u(0) = u_0 \in \mathbb{R}^n$.

How to solve? Diagonalize A!

Eg: $u' = Au$ $\quad A = \frac{1}{10}\begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix}$ $\quad u(0) = u_0 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

Diagonalize: $\lambda_1 = 2$ $\quad w_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\quad \lambda_2 = 1/2$ $\quad w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Say $u(t) = y_1(t) w_1 + y_2(t) w_2$ (write sol'n in eigenbasis).

$u' = y_1' w_1 + y_2' w_2$ $\qquad Au = 2 y_1 w_1 + \frac{1}{2} y_2 w_2$

Want these to be equal:

want $y_1' = 2y_1$ $\quad y_2' = \frac{1}{2} y_2$.

Also $u_0 = x_1 w_1 + x_2 w_2 \Rightarrow x_1 = 1$ $\quad x_2 = 2$

$\Rightarrow u_0 = w_1 + 2w_2$ (write initial value in eigenbasis)

$u(0) = y_1(0) w_1 + y_2(0) w_2 \Rightarrow$ want $y_1(0) = 1$ $\quad y_2(0) = 2$

Now we have two 1-variable ODEs:

$y_1' = 2y_1$ $\quad y_1(0) = 1 \rightsquigarrow y_1 = 1 \cdot e^{2t}$

$y_2' = 1/2\, y_2$ $\quad y_2(0) = 2 \rightsquigarrow y_2 = 2 \cdot e^{1/2 t}$

$$\Rightarrow u(t) = e^{2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2e^{1/2 t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{2t} - 2e^{1/2 t} \\ 3e^{2t} + 2e^{1/2 t} \end{pmatrix}$$

Check: $u(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ ✓

$$u' = \begin{pmatrix} 4e^{2t} - e^{1/2 t} \\ 6e^{2t} + e^{1/2 t} \end{pmatrix}$$

equal ✓

$$Au = \frac{1}{10}\begin{pmatrix} 11(2e^{2t} - 2e^{1/2 t}) + 6(3e^{2t} + 2e^{1/2 t}) \\ 9(2e^{2t} - 2e^{1/2 t}) + 14(3e^{2t} + 2e^{1/2 t}) \end{pmatrix} = \begin{pmatrix} 4e^{2t} - e^{1/2 t} \\ 6e^{2t} + e^{1/2 t} \end{pmatrix}$$

Procedure for solving ODEs: To solve $u' = Au$, $u(0) = u_0$:

(1) Diagonalize A: eigenbasis $w_1, \dots, w_n$, eigenvals $\lambda_1, \dots, \lambda_n$

(2) Solve $u_0 = x_1 w_1 + \dots + x_n w_n$

(3) Answer: $\boxed{u(t) = x_1 e^{\lambda_1 t} w_1 + \dots + x_n e^{\lambda_n t} w_n}$

Check:

$$u(0) = x_1 w_1 + \cdots + x_n w_n = u_0 \quad \checkmark$$

$$u' = \lambda_1 x_1 e^{\lambda_1 t} w_1 + \cdots + \lambda_n x_n e^{\lambda_n t} w_n$$
$$= A(x_1 e^{\lambda_1 t} w_1 + \cdots + x_n e^{\lambda_n t} w_n) = Au \quad \checkmark$$

NB: Compare to solving difference equation $v_m = Av_{m-1} = A^m v_0$:

(1) Diagonalize A: eigenbasis $w_1, \ldots, w_n$, eigenvals $\lambda_1, \ldots, \lambda_n$

(2) Solve $v_0 = x_1 w_1 + \cdots + x_n w_n$

(3) Answer: $\boxed{v_m = x_1 \lambda_1^m w_1 + \cdots + x_n \lambda_n^m w_n}$

Not just an analogy: ODE is a "limit of difference equations"...

Also works with complex eigenvalues!

Eg: Hooke's law: $u(t) = \begin{pmatrix} p(t) \\ v(t) \end{pmatrix}$

$u' = Au \quad A = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \quad u(0) = \begin{pmatrix} p_0 \\ v_0 \end{pmatrix}$

$p(\lambda) = \lambda^2 + k \quad k > 0 \Rightarrow \lambda = i\sqrt{k}, \ \bar{\lambda} = -i\sqrt{k}$

$A - \lambda I_2 = \begin{pmatrix} -i\sqrt{k} & 1 \\ & \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1 \\ i\sqrt{k} \end{pmatrix} \quad \bar{v} = \begin{pmatrix} 1 \\ -i\sqrt{k} \end{pmatrix}$ eigenbasis

$\begin{pmatrix} p_0 \\ v_0 \end{pmatrix} = x_1 v + x_2 \bar{v} \xRightarrow[\text{hand}]{\text{by}} x_1 = \frac{1}{2}\left(p_0 - \frac{v_0}{\sqrt{k}} i\right) \quad x_2 = \frac{1}{2}\left(p_0 + \frac{v_0}{\sqrt{k}} i\right)$

$$\Rightarrow u(t) = \begin{pmatrix} p(t) \\ v(t) \end{pmatrix} = \frac{1}{2}\left(p_0 - \frac{v_0}{\sqrt{k}} i\right) e^{i\sqrt{k}t} \begin{pmatrix} 1 \\ i\sqrt{k} \end{pmatrix} + \frac{1}{2}\left(p_0 + \frac{v_0}{\sqrt{k}} i\right) e^{-i\sqrt{k}t} \begin{pmatrix} 1 \\ -i\sqrt{k} \end{pmatrix}$$

Get rid of imaginary numbers: solve for $p(t)$:

$$p(t) = \frac{1}{2}\left(p_0 - \frac{v_0}{\sqrt{k}} i\right) e^{i\sqrt{k}t} + \frac{1}{2}\left(p_0 + \frac{v_0}{\sqrt{k}} i\right) e^{-i\sqrt{k}t}$$

$$= \mathrm{Re}\left[\left(p_0 - \frac{v_0}{\sqrt{k}} i\right) e^{i\sqrt{k}t}\right] = \mathrm{Re}\left[\left(p_0 - \frac{v_0}{\sqrt{k}} i\right)\left(\cos\sqrt{k}t + i\sin\sqrt{k}t\right)\right]$$

$$p(t) = p_0 \cos(\sqrt{k}t) + \frac{v_0}{\sqrt{k}} \sin(\sqrt{k}t)$$

check:

$$p(0) = p_0 \checkmark$$

$$v(t) = p'(t) = -\sqrt{k}\, p_0 \sin(\sqrt{k}t) + v_0 \cos(\sqrt{k}t)$$

$$v(0) = v_0 \checkmark$$

$$p''(t) = v'(t) = -k\, p_0 \cos(\sqrt{k}t) - \sqrt{k}\, v_0 \sin(\sqrt{k}t)$$

$$= -k\, p(t) \checkmark$$