Symmetric Matrices

Let S be a (real) symmetric matrix: $S = S^T$.

Eigenvectors & eigenvalues? Eg: $S = A^TA$

Eg: $S = \frac{1}{9}\begin{pmatrix} 5 & -8 & 10 \\ -8 & 11 & 2 \\ 10 & 2 & 2 \end{pmatrix}$ Q: What do you notice about the eigenspaces?

Spectral Theorem:

A real symmetric matrix has an orthonormal eigenbasis of real eigenvectors:

$S = QDQ^T$ Q: orthogonal D: diagonal

$Q^T = Q^{-1}$

This means:

(1) Eigenvectors with different eigenvalues are orthogonal.

$\rightarrow$ Say $Sv = \lambda v$ $Sv' = \lambda' v'$ $\lambda \neq \lambda'$

- $v \cdot Sv' = v \cdot (\lambda' v') = \lambda' v \cdot v'$
- $v \cdot Sv' = v^T S v' = (S^T v)^T v' = (Sv)^T v' = \lambda v^T v' = \lambda v \cdot v'$

subtract: $0 = (\lambda' - \lambda)\, v \cdot v'$ $\lambda' - \lambda \neq 0 \Rightarrow v \cdot v' = 0$

(2) All eigenvalues are real

$\rightarrow$ Say $Sv = \lambda v \Rightarrow S\bar{v} = \bar{\lambda}\bar{v}$

- $\bar{v} \cdot Sv = \lambda(\bar{v} \cdot v)$ (as above)
- $\bar{v} \cdot Sv = \bar{\lambda}(\bar{v} \cdot v)$ (as above)

$\Rightarrow \lambda(\bar{v} \cdot v) = \bar{\lambda}(\bar{v} \cdot v)$ $\bar{v} \cdot v = \sum_i |v_i|^2 \neq 0 \Rightarrow \lambda = \bar{\lambda}$

$\lambda \in \mathbb{R}$
$\Uparrow$

(3) A symmetric (real) matrix is diagonalizable.

$\rightarrow$ Need abstract linear algebra.

Eg: $S = \frac{1}{9}\begin{pmatrix} 5 & -8 & 10 \\ -8 & 11 & 2 \\ 10 & 2 & 2 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-1)(\lambda+1)(\lambda-2)$

Compute $\mathrm{Nul}(A - \lambda_i I_3)$ ... orthogonal

normalize

$\lambda_1 = 1 \quad w_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad \lambda_2 = -1 \quad w_2 = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \qquad \lambda_3 = 2 \quad w_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$$\Rightarrow S = QDQ^T \qquad Q = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Procedure to Orthogonally Diagonalize a Symmetric Matrix:

(1) Diagonalize S (this will succeed).

(2) Normalize your eigenvectors / run Gram-Schmidt if $GM \geq 2$.

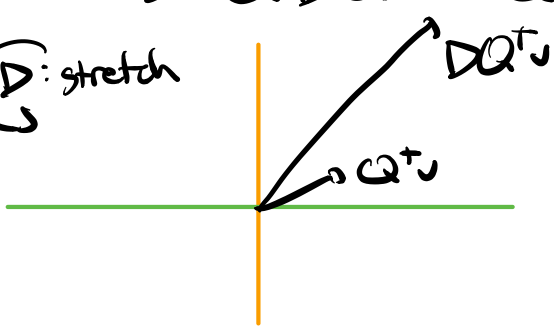
(3) Put them all together $\rightarrow$ orthonormal eigenbasis!

Eg: $S = \frac{1}{2}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\lambda_1 = 2 \quad w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = 3 \quad w_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

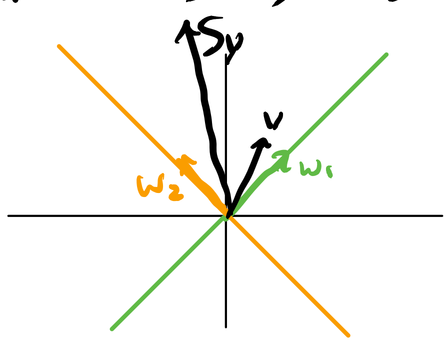
$$S = QDQ^T \qquad Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

D: stretch



Q^T: rotate CW by $45°$

Q: rotate CCW by $45°$



Same picture as before, but simpler: Q, Q^T preserve lengths & angles.

Exercise: Outer product form of $S = QDQ^T$:

$$S = \lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T \qquad Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

compare: $P_V = QQ^T \longleftrightarrow P_V = q_1 q_1^T + \cdots + q_d q_d^T$

Application: Constrained Optimization.

Def: A quadratic form is a function

$q(x_1, \ldots, x_n) =$ sum of terms of the form $a_{ij} x_i x_j$

$\longrightarrow$ eg $q(x_1, x_2) = \frac{5}{2} x_1^2 + \frac{5}{2} x_2^2 - x_1 x_2$ ← cross term

A quadratic form is diagonal if there are no cross terms:

$q(x_1, \ldots, x_n) = a_{11} x_1^2 + \cdots + a_{nn} x_n^2$

$\longrightarrow$ eg $q(x_1, x_2) = 2x_1^2 + 3x_2^2$

NB: $q(ax_1, \ldots, ax_n) = a^2 q(x_1, \ldots, x_n)$

Quadratic Optimization Problem:

Find min/max values of $q(x_1, \ldots, x_n)$ subject to the constraint $x_1^2 + \cdots + x_n^2 = 1$ ie. $\|x\| = 1$

Equivalently: (if $q(x) > 0$ for $x \neq 0$)

Find min/max value of $x_1^2 + \cdots + x_n^2 = \|x\|^2$ subject to the constraint $q(x_1, \ldots, x_n) = 1$

Why are these the same?

if $A>0$

$$q(x)=1 \iff q\left(\frac{x}{\|x\|}\right)=\frac{1}{\|x\|^2} \quad \& \quad q(x)=A \iff q\left(\frac{x}{\sqrt{A}}\right)=1$$

shortest x s.t. $q(x)=1 \longleftrightarrow$ max value of $q(x/\|x\|)$

longest x s.t. $q(x)=1 \longleftrightarrow$ min value of $q(x/\|x\|)$

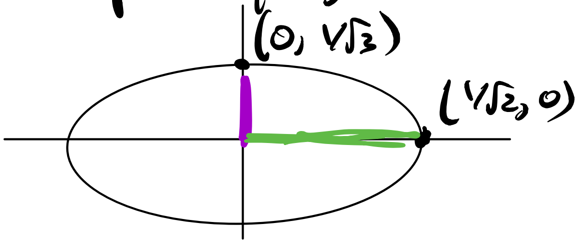
Arises in: Lagrangian mechanics ($q=$ kinetic energy), finding minima/maxima of functions (q comes from Hessian matrix), ...

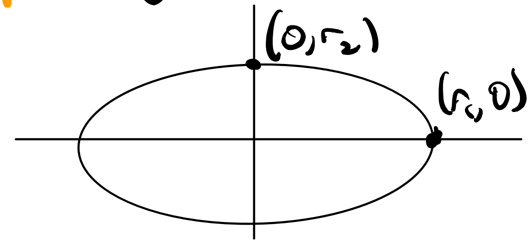
Easy case: q is diagonal

Eg: $q(x_1,x_2)=2x_1^2+3x_2^2$

Let's plot $q(x_1,x_2)=1$:

ellipse: $\left(\frac{x_1}{r_1}\right)^2+\left(\frac{x_2}{r_2}\right)^2=1$





Shortest x: $\pm\frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\end{pmatrix}$

Max value: $1/\|x\|^2=3$

Longest x: $\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$

Min value: $1/\|x\|^2 : 2$

What if q is not diagonal?

Eg: $q(x_1,x_2)=\frac{5}{2}(x_1^2+x_2^2)-x_1x_2 \leftarrow$ cross term

Relation to Symmetric Matrices:

$$q(x_1, \dots, x_n) = \sum a_{ij} x_i x_j = x^T S x \quad \text{for}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad S = \frac{1}{2}\left[\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{pmatrix}^T \right]$$

In our case: $q(x_1, x_2) = x^T S x$

$$S = \frac{1}{2}\left[\begin{pmatrix} 5 & -1 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ -1 & 5 \end{pmatrix} \right] = \frac{1}{2}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

check: $$(x_1 \ x_2) \cdot \frac{1}{2}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 5x_1 - x_2 \\ -x_1 + 5x_2 \end{pmatrix}$$

$$= \frac{1}{2}(5x_1^2 - x_1 x_2 - x_2 x_1 + 5x_2^2) \checkmark$$

Orthogonally Diagonalize!

$$S = Q D Q^T \quad Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

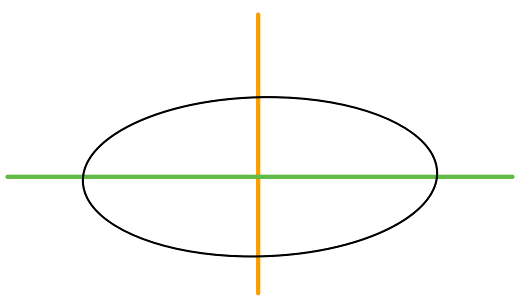
Change Coordinates: $y = Q^T x \quad y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) \quad y_2 = \frac{1}{\sqrt{2}}(-x_1 + x_2)$

$$\Rightarrow q(x) = x^T Q D Q^T x = (Q^T x)^T D (Q^T x) = y^T D y$$

$$= (y_1 \ y_2)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \cdot \begin{pmatrix} 2y_1 \\ 3y_2 \end{pmatrix} = 2y_1^2 + 3y_2^2$$

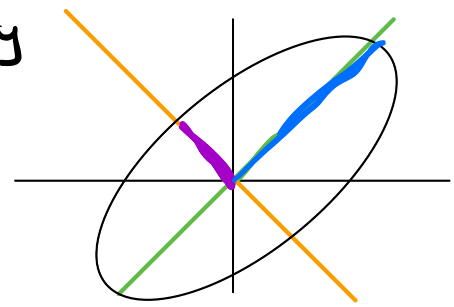
no cross terms

Picture:



$y = Q^T x \quad x = Qy$

$\xrightarrow{Q}$



$$2y_1^2 + 3y_2^2 = 1 \xrightarrow[\substack{y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) \\ y_2 = \frac{1}{\sqrt{2}}(-x_1 + x_2)}]{\text{substitute}} \frac{5}{2}(x_1^2 + x_2^2) - x_1 x_2 = 1$$

Answer:

Shortest x: $\frac{1}{\sqrt{3}} \omega_2$

Max value: 3

Longest x: $\frac{1}{\sqrt{2}} \omega_1$

Min value: 2