Basic Definitions

Scalars: a (real) number (later complex)

Notation: $c \in \mathbb{R}$

Eg: $c = 7, -\pi, 2^e, 0, \ldots$

Vectors: a list of real numbers

Notn: $v \in \mathbb{R}^n$ $n =$ length of the vector

$=$ size / dimension

Eg: $\begin{pmatrix} 7 \\ -\pi \\ 2^e \end{pmatrix} \in \mathbb{R}^3$

Note: Usually write as a column

Can also write $(7, -\pi, 2^e)$

Two vectors are **equal** if they have the same size and all components are equal.

Zero vector: $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

I write vectors with normal letters: u, v, w

Often see: $\vec{u}, \vec{v}, \vec{w}$

or: $\mathbf{u}\ \mathbf{v}\ \mathbf{w}$

Scalar Multiplication: scalar × vector:

$$c \in \mathbb{R} \quad v \in \mathbb{R}^n \rightsquigarrow cv \in \mathbb{R}^n$$

$$c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \qquad 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

Vector + / −: $v, w \in \mathbb{R}^n \rightsquigarrow v+w,\ v-w \in \mathbb{R}^n$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \pm \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \pm y_1 \\ \vdots \\ x_n \pm y_n \end{pmatrix}$$

$\rightsquigarrow$ only makes sense for vectors of the same size!

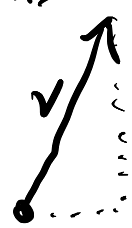
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$$

Geometry

Draw a vector as an arrow.

Components are displacements in different directions.

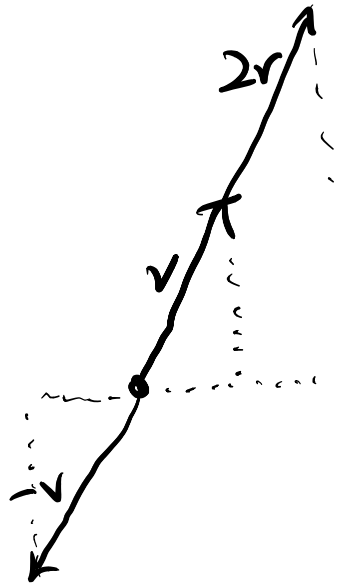
$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Scalar multiplication:

- changes the length, can reverse direction

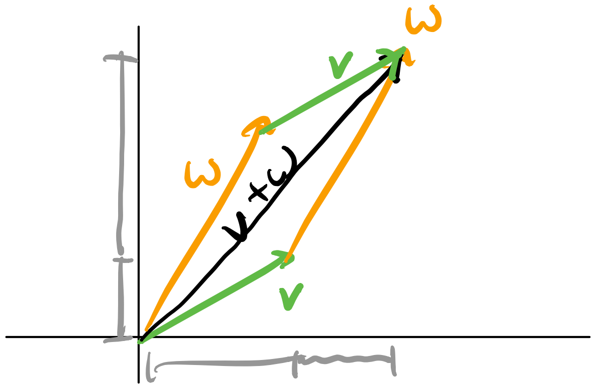
$v=\begin{pmatrix}1\\2\end{pmatrix}$



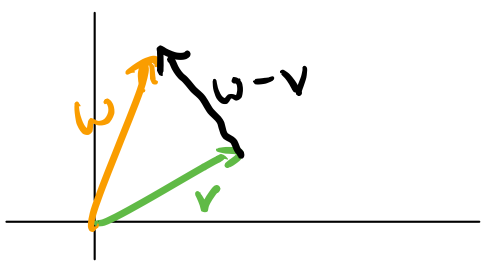
$2v=\begin{pmatrix}2\\4\end{pmatrix}$

$-v=\begin{pmatrix}-1\\-2\end{pmatrix}$

Vector addition: Parallelogram law



Vector Subtraction: $v+(w-v)=w$

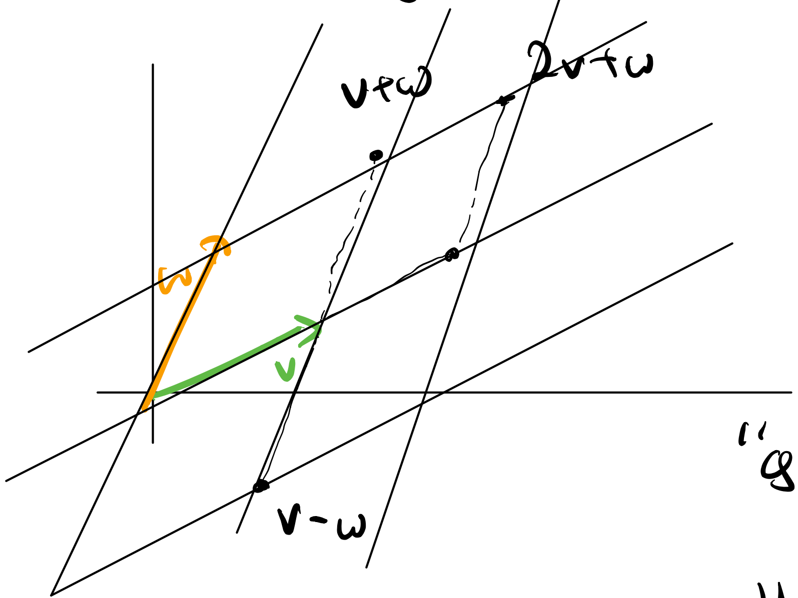


vector from head of v
to head of w

Linear Combinations: combine addition & scalar mult.

$$v,w\in\mathbb{R}^n \quad c,d\in\mathbb{R} \quad \rightsquigarrow \quad cv+dw\in\mathbb{R}^n$$

$$c\begin{pmatrix}x_1\\x_2\end{pmatrix}+d\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}cx_1+dy_1\\cx_2+dy_2\end{pmatrix}$$



$2v+w$

"$cv+dw$":

"go $c\times$ length of v in v-direction then $d\times$ length of w in w-direction"

Dot Product:

$$\overbrace{\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\cdot\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}}^{\text{same size}}=x_1y_1+\cdots+x_ny_n\in\mathbb{R}$$

vector $\times$ vector $\rightsquigarrow$ scalar

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix}=1\cdot2+2\cdot1+3\cdot(-3)=-5$$

Rules for vector algebra:

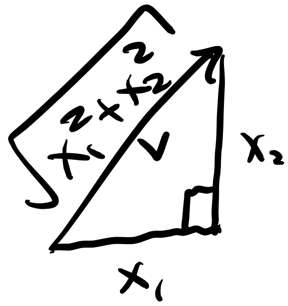
- $u\cdot(cw)=c\cdot(u\cdot w)=(cu)\cdot w$
- $u\cdot v=v\cdot u$
- $u\cdot(v+w)=u\cdot v+u\cdot w$
- $\cancel{u\cdot(v\cdot w)=(u\cdot v)\cdot w}$

$v\cdot w$ is a <u>scalar</u>

Special Case:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1x_1 + x_2x_2 = x_1^2 + x_2^2 \geq 0$$

In general, $v \cdot v$ = sum of squares of entries ≥ 0



Length of v is $\sqrt{v \cdot v} = \|v\|$

$$\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| = \sqrt{x^2+y^2+z^2}$$

Sanity Check: $c \in \mathbb{R}$ $v \in \mathbb{R}^n$

$$\|c \cdot v\|^2 = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \cdot \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

$$= (cx_1)(cx_1) + \cdots + (cx_n)(cx_n)$$

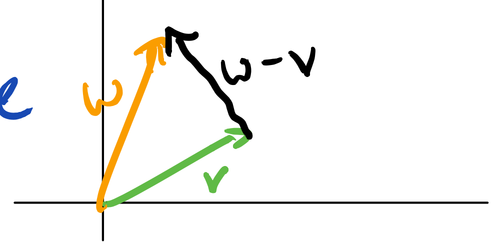
$$= c^2(x_1^2 + \cdots + x_n^2) = c^2 \|v\|^2$$

$$\Rightarrow \|c \cdot v\| = |c| \cdot \|v\|$$

$\leadsto$ $2v$ is twice as long as v

So is $-2v$

Distance



distance from v to w $:= \|w-v\|$

$= \|v-w\|$

Unit Vector: a vector v with $\|v\|=1$

i.e. vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ s.t. $x_1^2+\cdots+x_n^2=1$

If $v \neq 0$, the unit vector in direction of v

is $u = \dfrac{v}{\|v\|}$

NB: $\|u\| = \left\| \dfrac{v}{\|v\|} \right\| = \dfrac{1}{\|v\|} \cdot \|v\| = 1$

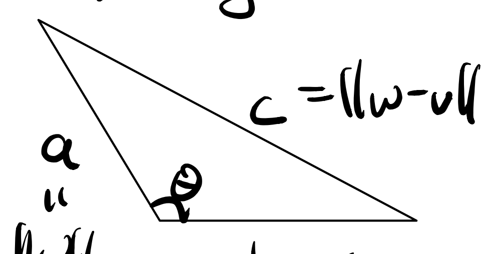
Eg: $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\|v\| = \sqrt{3^2+4^2} = 5$

$u = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

Angles: what is $v \cdot w$ if $w \neq v$? (geometrically)

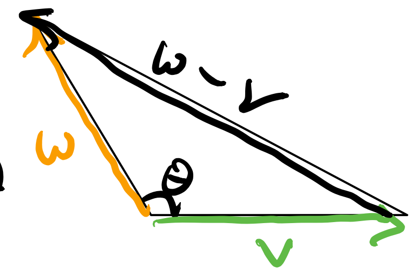
Law of Cosines:

$c^2 = a^2 + b^2 - 2ab \cos\theta$



$\|w\|$ $\quad b = \|v\|$

Vector version:



$$\|w - v\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\cdot\|w\| \cos\theta$$

$$(w-v)\cdot(w-v) = w\cdot w - 2v\cdot w + v\cdot v$$
$$= \cancel{\|w\|^2} + \cancel{\|v\|^2} - 2v\cdot w$$
$$= \cancel{\|v\|^2} + \cancel{\|w\|^2} - 2\|v\|\cdot\|w\|\cos\theta$$

$$\Longrightarrow \quad v\cdot w = \|v\|\cdot\|w\| \cdot \cos\theta$$

$$\boxed{\cos(\theta) = \frac{v\cdot w}{\|v\|\cdot\|w\|}} \qquad v, w \neq 0$$

In $\dim \geq 3$ this defines

angle from v to w = $\cos^{-1}\left(\dfrac{v\cdot w}{\|v\|\cdot\|w\|}\right)$

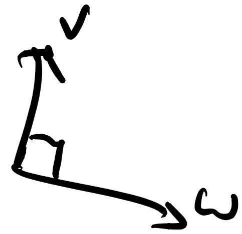
$$|v\cdot w| = \|v\|\cdot\|w\|\,|\cos\theta| \leq \|v\|\cdot\|w\|$$

Schwartz Inequality:

$$|v\cdot w| \leq \|v\|\cdot\|w\|$$

Defn: Two vectors are orthogonal or perpendicular if $v\cdot w = 0$

Says $\cos(\theta) = 0 \implies \theta = \pm 90^\circ$



A matrix is a box holding a grid of numbers

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$m = 3$ rows

$n = 2$ columns

● = (3,2) entry: 3^{rd} row, 2^{nd} col

$m \times n = 3 \times 2 =$ size of the matrix

Addition & Scalar multiplication componentwise:

$$c \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} c & 4c \\ 2c & 5c \\ 3c & 6c \end{bmatrix} \qquad \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 10 & 16 \\ 12 & 18 \end{bmatrix}$$

same size!

NB: A vector is a matrix with 1 column!

row vector: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Matrix × vector: 2 ways

- column first:

$m=3$
$n=2$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

!! The linear combination of columns of the matrix w/ scalars x_1, x_2

same

- Row first:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1\cdot x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix}$$

!! entries of product are dot products w/ the rows

NB: Only makes sense if # cols of matrix = size of the vector!

$(m \times n)\cdot(n \times 1) \rightsquigarrow m \times 1$

NB: Recover $v \cdot w$ by thinking of v as a row vector:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_2 + x_3y_3 \end{bmatrix}$$

Matrix × Matrix: also 2 ways

- By columns:

$$A\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix} := \begin{bmatrix} | & | & | \\ Au & Av & Aw \\ | & | & | \end{bmatrix}$$

- By rows:

$$\begin{bmatrix} — & x & — \\ — & y & — \end{bmatrix}\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix} = \begin{bmatrix} x\cdot u & x\cdot v & x\cdot w \\ y\cdot u & y\cdot v & y\cdot w \end{bmatrix}$$

NB: $(m\times n)\times(n\times p) \rightsquigarrow (m\times p)$

same

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{bmatrix}$$

$(2\times\cancel{3})$ $(\cancel{3}\times 2)$

- cols:

$$\left[\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 17 & 2 \\ \cancel{-15} & \cancel{4} \end{bmatrix} \quad (2\times 2)$$

-13 $\quad 3$

- rows:

$$\begin{bmatrix} 1 & 2 & 3 \\ \ldots \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ \ldots \end{bmatrix} = \begin{bmatrix} 17 & 2 \\ \ldots \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -15 & 4 \end{bmatrix}$$

Eg: Identity matrix   square $(m=n)$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Check: $I_n A = A = A I_n$

Properties: ASSUME DIMENSIONS MATCH

- Associativity: $(AB)C = A(BC)$

  $\rightsquigarrow$ write ABC

  $\to$ $C = v$ vector: $(AB)v = A(Bv)$

- Distributivity:

  $A(B+C) = AB + AC \qquad (A+B)C = AC + BC$

- Commutativity fails!

  $AB \neq BA \qquad A: 3\times 2 \quad B: 2\times 4$

  AB ✓ BA ✗

  even for 2×2 matrices!

- Cancellation fails!

$$AB = AC \quad A \neq 0 \quad \not\Rightarrow \quad B = C$$