3 Ways to Write a Linear System

(1) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

As a system of linear equations

(2) $$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \\ 3x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}}_{b}$$

This is the matrix equation $Ax = b$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

is the coefficient matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ is the (unknown variable) vector

NB: A is an $m \times n$ matrix where

m = # equations

n = # variables

$x \in \mathbb{R}^n$ = all vectors of size n

$b \in \mathbb{R}^m$

(3) Squash A & b together:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \qquad [A \mid b]$$

this is an augmented matrix

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

How to solve systems of linear equations?

2 methods:

- elimination
- ~~substitution~~

Elimination: simplify equations so first variable appears only in first equation

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_2 -= 2R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\xrightarrow{R_3 -= 3R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned}$$

eliminated x_1!

2 eqns in 2 vars

$$R_3 -= \frac{7}{5} R_2 \longrightarrow \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -\tfrac{50}{7}x_3 &= -\tfrac{150}{7} \end{aligned}$$

Solve by back-substitution:

$$-\frac{50}{7}x_3 = -\frac{150}{7} \implies x_3 = 3$$

$$2 = -7x_2 - 4x_3 = -7x_2 - 12 \implies -7x_2 = 14$$

$$x_2 = -2$$

$$\begin{aligned} 6 &= x_1 + 2x_2 + 3x_3 \\ &= x_1 - 4 + 9 = x_1 + 5 \implies x_1 = 1 \end{aligned}$$

Check: $1 - 4 + 9 = 6$ ✓

$2 + 6 + 6 = 14$ ✓

$3 - 2 - 3 = -2$ ✓

Q: What do I do here?

$$\begin{aligned} -7x_2 - 4x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

swap order

Legal row operations:

(1)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_2 -= 2R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

row replacement

replace R_2 by $R_2 - 2R_1$

(2)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_2} \begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 14 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

row swap

(3)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_1 \times = 2} \begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 12 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

scalar multiplication

Reversible! Have same solutions! (whole point)

[Augmented] Matrix Form

(1)
$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow[\text{row replacement}]{R_2 -= 2R_1} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{bmatrix}$$

(2) $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow[\text{row swap}]{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -3 & 2 & 14 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & -1 & -2 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow[\text{scalar } \times]{R_1 \times= 2} \begin{bmatrix} 2 & 4 & 6 & 12 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix}$

Solve in matrix form:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow{R_2 -= 2R_1} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 -= 3R_1} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{bmatrix}$$

$$\xrightarrow{R_3 -= \frac{5}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$x_1 + 2x_2 + 3x_3 = 6$

$-7x_2 - 4x_3 = 2$

$-\frac{50}{7}x_3 = -\frac{150}{7}$

Another Example

$x_1 + 2x_2 - x_3 = 4$

$2x_1 + 4x_2 + x_3 = 4$

$$\leadsto \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 4 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 -= 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right]$$

Done? Yes!

$$x_1 + 2x_2 - x_3 = 4$$
$$3x_3 = 12$$

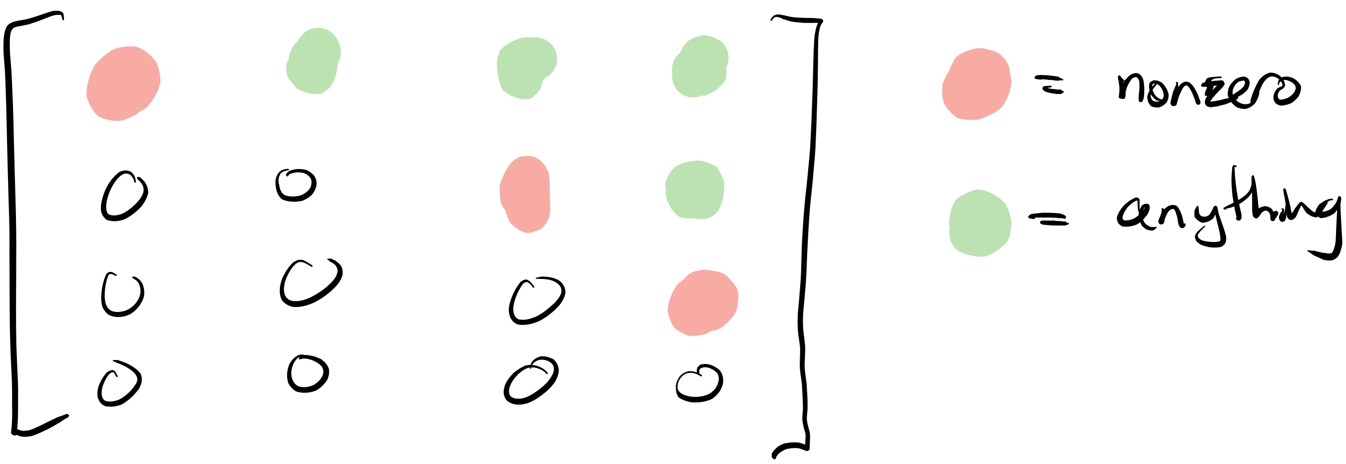
$x_3 = 4$ $\quad x_1 + 2x_2 - 4 = 4$

$$x_1 = 8 - 2x_2$$

x_2 is any real number

What does it mean to be "done"
(in matrix form)

Def: A matrix is in row echelon form (REF) if
(1) The first nonzero entry of each row is to the right of the row above it
(2) All zero rows are at the bottom



$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

ignore augmentation line

Solving a linear system $\equiv$ Putting a matrix into REF using row operations

Def: The pivot positions (pivots) of a matrix are the 1st nonzero entries of each row after you put it into REF.

● = pivot positions