Recall: The pivot positions (pivots) of a matrix are the 1st nonzero entries of each row after putting it into REF.

Q: In an augmented matrix, what if there's a pivot in the last column?

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] \longleftrightarrow \begin{aligned} x + 2y &= 4 \\ y &= 3 \\ 0 &= 1 \end{aligned}$$

!! no solutions!

Def: A system is consistent if it has a solution. inconsistent otherwise.

(inconsistent $\Longleftrightarrow$ pivot in last column of $[A|b]$)

Gaussian Elimination
or How a Computer Solves a Linear System (almost)

Algorithm (Gaussian Elimination)

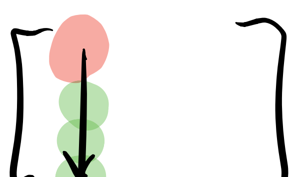
Input: a matrix

Output: an equivalent matrix in REF

↳ obtained by doing row ops

(a) Row swap so the 1st column with a nonzero entry has a nonzero entry in 1st row.

(b) Row replace to kill all entries below that entry.



now ignore 1st row & 1st column

Recurse until done.

Eg:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 -= 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 -= R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

no pivot!

$$x_1 + x_2 - x_3 = 3$$
$$x_2 + x_3 = 4$$
$$x_3 = 5$$

Jordan Substitution

→ Basically does back substitution

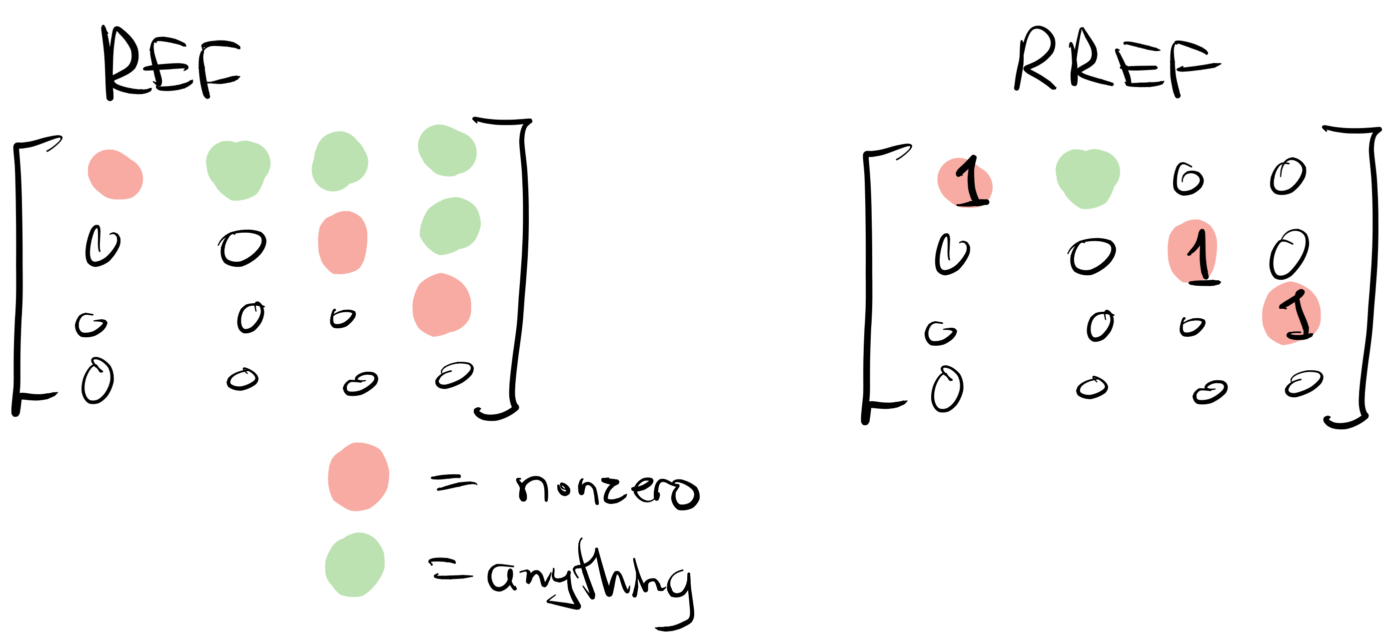
→ Very useful when many solutions

Def: A matrix is in reduced row echelon form if

(1–2) It's in REF

(3) All pivots are 1

(4) A pivot is the only nonzero entry in its column.



Eg:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}$$

Q: How to put a matrix in RREF?

Algorithm (Jordan Substitution):
Input: A matrix in REF
Output: An equivalent matrix in RREF.
Loop (starting at last pivot)
(a) Scale that row so pivot = 1
(b) Use row replacements to kill entries above.

Thm: RREF is <u>unique</u>
→ As long as you do legal row ops, there's only one matrix in RREF that is equivalent to a starting matrix.

Elimination Using Matrices

Def: The nxn identity matrix is
$$I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

NB: $I_n A = A \qquad B I_n = B$

Def: An ~~elimination~~ elementary matrix is a matrix obtained from I_n by doing <u>one</u> row op.

Eg:

- $R_1 += 2R_2$ $\qquad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $R_2 \times= 2$ $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $R_1 \leftrightarrow R_2$ $\qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Fact: If E is an $m \times m$ elementary matrix
A is a matrix with m rows
$\Rightarrow EA$ performs that row operation on A

☆ (row operations) $\longleftrightarrow$ (mult. with elementary mtx)

Eg:

$R_1 += 2R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 += 2R_2} \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Multiple row ops:

$$A \xrightarrow{R_1 \mathrel{+}= 2R_2} E_1 A \xrightarrow{R_3 \mathrel{\times}= 2} E_2(E_1 A) \xrightarrow{R_2 \leftrightarrow R_3} E_3(E_2(E_1 A))$$

$$= E_3 E_2 E_1 A$$

Suppose RREF(A) is I_n

$$I_n = E_r \cdots E_3 E_2 E_1 A = \overbrace{(E_r \cdots E_3 E_2 E_1)}^{A^{-1}} A$$

Def: An $n \times n$ (square!) matrix is invertible if there exists another $n \times n$ matrix B such that

$$AB = I_n = BA$$

Notation: $B = A^{-1}$

B is the inverse of A.

NB: "Left-inverse" = "right inverse":

$$AB = I_n \qquad CA = I_n$$

$$B = (CA)B = CAB = C(AB) = CI_n = C$$

$$\Rightarrow B = C$$

So, if RREF(A) is I_n

then $A^{-1} = E_r E_{r-1} \cdots E_2 E_1$

$= E_r E_{r-1} \cdots E_2 E_1 I_n$

This is the matrix you get by doing the same row ops to I_n.

Algorithm (Matrix Inversion)

Input: a square matrix

Output: the inverse, or "not invertible"

- Perform Gauss-Jordan on $[A \mid I_n]$. If you get $[I_n \mid B]$ then $B = A^{-1}$. Otherwise A is not invertible.

Eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = ?$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 -= 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \div= -2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\xrightarrow{R_1 -= 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Check: $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Why? Suppose A is invertible.

Solve $Ax = b$ for 1M values of b.

$$Ax = B \iff A^{-1}(Ax) = A^{-1}b$$

$$(A^{-1}A)x = A^{-1}b$$

$$I_n x = A^{-1}b$$

$$Ax = b \iff x = A^{-1}b$$

Formula for 2x2 inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad-bc=0 \implies$ not invertible.