Inverse Matrix:

An $n \times n$ matrix A is invertible if there's another matrix A^{-1} s.t. $AA^{-1} = I_n = A^{-1}A$.

[only check one]

This happens when $\text{RREF}(A) = I_n$.

$$\underbrace{(E_r E_{r-1} \cdots E_1)}_{\text{elem. matrices}} A = I_n$$

Q: How many pivots?

Thm!! An $n \times n$ matrix is invertible $\Longleftrightarrow$ it has n pivots.

($\Longrightarrow$) "full column rank" (later)

Solving equations:

$$\boxed{Ax = b \quad \Longleftrightarrow \quad x = A^{-1}b}$$

Computation: $[A \mid I_n] \overset{\text{RREF}}{\rightsquigarrow} [I_n \mid A^{-1}]$

Formula for 2x2 inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0 \Longrightarrow$ not invertible.

Elementary Matrices:

Row operation $\rightsquigarrow$ E

Undo row operation $\rightsquigarrow$ E^{-1}

$$R_1 += 2R_2 \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

Undo

$$R_1 -= 2R_2 \rightsquigarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E^{-1}$$

Properties:

(1) If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$.

$\longrightarrow$ $AA^{-1} = I_n \Rightarrow (A^{-1})^{-1} = A$

(2) If A & B are invertible then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

$\longrightarrow$ Why not $A^{-1}B^{-1}$?

$$AB \cdot B^{-1}A^{-1} = A(\underbrace{BB^{-1}}_{I_n})A^{-1} = AI_nA^{-1}$$
$$= AA^{-1} = I_n$$

$$ABA^{-1}B^{-1} = ??$$

LU Decomposition, or
How a Computer solves a Linear System (almost!)

Gaussian Elimination on an $n \times n$ matrix takes:

$$\underbrace{\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}}_{n} \left.\vphantom{\begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix}}\right\} n-1$$

1st column:

$$\begin{array}{rl} n(n-1) & \text{mult} \\ + \; n(n-1) & \text{add} \\ \hline 2n(n-1) & \text{flops} \end{array}$$

"floating point ops"

2nd column: $2(n-1)(n-2)$

Total: $2n(n-1) + 2(n-1)(n-2) + \cdots + 2 \cdot 2 \cdot 1$

$$= \frac{2n(n+1)(n-1)}{3} \approx \frac{2}{3} n^3 \text{ flops}$$

Back-Substitution

$$\begin{array}{rr} & \vdots \\ \bullet x_{n-2} + \bullet x_{n-1} + \bullet x_n = \bullet & \\ \bullet x_{n-1} + \bullet x_n = \bullet & \\ \bullet x_n = \bullet & \end{array}$$

$$\begin{array}{rl} n & \text{flops} \\ \vdots & \\ 3 & \text{flops} \\ 2 & \text{flops} \\ + \; 1 & \text{flop} \\ \hline \frac{n(n+1)}{2} & \approx \frac{1}{2} n^2 \text{ flops} \end{array}$$

$\frac{2}{3}n^3$ is a lot! $n = 1000 \rightarrow n^3 = 1{,}000{,}000{,}000$

What if you need to solve $Ax = b$ for 1,000,000 values of b? Don't want to eliminate every time!

Def: A matrix is **upper / lower triangular** if all entries below/above the diagonal are 0.

upper: $\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix}$ lower: $\begin{bmatrix} \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{bmatrix}$

Unitriangular: triangular and diagonal entries $= 1$.

Eg: REF $\Rightarrow$ upper-Δ

Fact: A product of square upper/lower (uni) triangular matrices is upper/lower (uni) triangular. Likewise w/ inverses.

Suppose $A = LU$ where

- L is uni lower-Δ
- U is upper Δ.

To solve $Ax = b$, two steps:

(1) Solve $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ \bullet & 1 & 0 \\ \bullet & \bullet & 1 \end{bmatrix} y = b \qquad \begin{aligned} y_1 &= b_1 \\ y_2 + \bullet\, y_1 &= b_2 \\ y_3 + \bullet\, y_2 + \bullet\, y_1 &= b_3 \end{aligned}$$

Solve using forward-substitution: $\approx \frac{1}{2}n^2$ flops.

(2) Solve $Ux = y$: backwards substitution $\approx \frac{1}{2}n^2$ flops

Total flops: $\approx n^2$ Way faster!

We're done!

$$Ax = LUx = L(Ux) = Ly = b$$

Recall: Row operation $\equiv$ left-mult by elementary matrix

Suppose A can be reduced to REF w/o row swaps.
So all row operations are row-replacements down

$$R_2 -= 2R_1 \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

lower-uni Δ

$$U = \underbrace{E_r \cdots E_2 E_1}_{\text{lower uni-}\Delta} A \implies A = \underbrace{(E_r \cdots E_2 E_1)^{-1}}_{\text{lower uni-}\Delta} U = LU$$

In this case, we can factor $A = LU$

↓

Write a matrix as a product of simpler matrices.
→ do lots of these

Still have to do elimination once, but:

- $\approx \frac{2}{3}n^3$ flops to compute $A = LU$
- Now only $\approx n^2$ flops to solve $Ax = b$ for each b.

Eg: A is 1000×1000, solve $Ax=b$ for $1{,}000{,}000$ b

- Without $A=LU$: $\approx (1000)^3\cdot(1000)^2 \approx 10^{15}$ flops
- With $A=LU$: $\approx (1000)^3 + (1000)^2\cdot(1000)^2$
  $\approx 10^{12}$ flops
  $\Rightarrow$ 1,000 times faster!

Algorithm (LU factorization)

Input: A matrix that can be reduced to REF w/o swaps

Output: A factorization $A=LU$

- U is REF
- L is square $m\times m$ lower-uni$\triangle$

- 1st column of L = 1st col of A $\div$ 1st pivot ($\neq 0$)
- Now eliminate 1st col of A
- Recurse.

Eg:

$$A=\begin{bmatrix} 2 & 4 & -1 & 5 & -2\\ -4 & -5 & 3 & -8 & 1\\ 2 & -5 & -4 & 1 & 8\\ -6 & 0 & 7 & -3 & 1\end{bmatrix} \xrightarrow[\text{1st col}]{\text{eliminate}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2\\ 0 & 3 & 1 & 2 & -3\\ 0 & -9 & -3 & -4 & 10\\ 0 & 12 & 4 & 12 & -5\end{bmatrix}$$

$$\xrightarrow[\text{2nd col}]{\text{eliminate}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2\\ 0 & 3 & 1 & 2 & -3\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 4 & 7\end{bmatrix} \xrightarrow[\text{4th col}]{\text{eliminate}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2\\ 0 & 3 & 1 & 2 & -3\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 5\end{bmatrix}$$

$\div 2$ $\quad$ $\div 3$ $\quad$ $\div 2$

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 1 & -3 & 1 & 0\\ -3 & 4 & 2 & 1\end{bmatrix}$$

$$\begin{array}{cc} \shortparallel & \shortparallel \\ L & U \end{array}$$

Check: $A = LU$

NB: If A is invertible, could compute A^{-1}, then $x = A^{-1}b$ each time.

- Computing A^{-1} takes 2x as long
- Less numerically accurate
- Solving $LUx = b$ $\sim n^2$ flops
  multiplying $A^{-1}x = b$ $\sim n^2$ flops
  $\Rightarrow$ no advantage to A^{-1}!

Inverses still useful, e.g. to explain why LU works.