Systems with Many Solutions

Eg:
$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right]$$

$$\overset{\text{RREF}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \rightsquigarrow \begin{aligned} x + 5z &= 1 \\ y + 2z &= -1 \end{aligned}$$

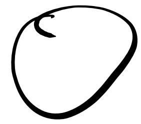
Observation: For every value of z, can solve for x & y:

$$x = 1 - 5z \qquad \text{or} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix}$$
$$y = -1 - 2z$$

This is the parametric form of the solution.
z is the free variable/parameter.

Implicit vs Parameterized Form

$x^2 + y^2 = 1$



$(\cos\theta, \sin\theta)$
$\theta \in [0, 2\pi]$

The system of equations
$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned}$$
are implicit equations of a line in $\mathbb{R}^3$. The parametric form
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix}$$
is a parametric equation for the same line.

$\rightarrow$ allows you to produce all points on the line by plugging in values for z.

Def: A **pivot column** of a matrix is a column with a pivot.

Def: A **free variable** in a system of equations is a variable whose column (in the coeff matrix) is not a pivot column.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

x, y not free

z is free

Procedure (Parametric Form)

To find the parametric form of the solutions of $Ax=b$:

(1) Put $(A|b)$ into RREF.

Stop if inconsistent (zero solns)

(2) Write out the corresponding equations.

(3) Move free variables to the right-hand side.

All solutions can be obtained by substituting any values for the free variables.

Eg: $x+y+z=1 \rightsquigarrow \begin{array}{cccc} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right] \\ x \quad y \quad z \end{array}$

free

$\rightsquigarrow x+y+z=1$

$\rightsquigarrow x = 1-y-z.$

parameters

Parameterized a plane with $\underline{2}$ parameters.

NB: if there $\underline{\text{is}}$ a free variable
$\Longrightarrow \infty$ solutions (substitute any value)

Thm: The equation $Ax=b$ has:
(1) Zero solutions if $(A|b)$ has a pivot in the last column (inconsistent)
(2) Exactly one solution if every col of $(A|b)$ $\underline{\text{except}}$ the last one has a pivot.
(3) ∞ solutions if the last col of $(A|b)$ & some other col has no pivot

(1) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 1 \end{array}\right]$ $0=1$ ✗

(2) $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$ $x=2$, $y=3$

(3) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ z is free!

The existence of a free variable only depends on A, not b.

Def: A matrix has full column rank if it has a pivot in every column.

Thm: If A has full col rank then $Ax=b$ has zero or one solutions for any b.
Otherwise, $Ax=b$ has 0 or ∞ solns for every b.

full col rank

RREF $\rightsquigarrow$ $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{array}\right]$

$c=0$: $x=a, y=b$ only soln

$c\neq 0$: no solns

not full col rank

RREF $\rightsquigarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 0 & c \end{array}\right]$

$c=0$: z free $\Rightarrow \infty$ solns

$c\neq 0$: no solns

The Column Picture

So far we have considered all x s.t. $Ax=b$ $\quad x\in\mathbb{R}^n$ (#cols)
$\rightarrow$ called the row picture (rows $\leftrightarrow$ equations)
Now we'll try to draw all b s.t. $Ax=b$ has a soln
$\rightarrow$ called the column picture $\quad b\in\mathbb{R}^m \leftarrow$ #rows

Def: A vector equation is an equation involving a linear combination of vectors w/unknown coefficients.

$$x_1\begin{pmatrix}1\\2\\6\end{pmatrix}+x_2\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix} \longleftarrow \text{geometry}$$

Column defn of matrix multiplication:

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \qquad Ax=b$$

algebra

Four Ways to Write a System of Equations

(1) Linear system

$$\begin{aligned} x_1 - x_2 &= 8 \\ 2x_1 - 2x_2 &= 16 \\ 6x_1 - x_2 &= 3 \end{aligned}$$

(2) Matrix equation

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(3) Augmented Matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

(4) Vector Equation

$$x_1\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Important Observation:

$$x_1\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$ has a solution (consistent)

$$\Longleftrightarrow \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \text{ is a LC of } \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \,\&\, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

In this case the solution is the vector of coefficients $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

This example: $$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$ → on purple plane

$$\Longrightarrow \text{solution is } \begin{pmatrix} -1 \\ -9 \end{pmatrix}$$

If we take $b = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \rightsquigarrow$ no solution.

↳ not on purple plane