Column Picture

vector equation

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \rightsquigarrow x_1\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Solving $Ax = b$ $\leftrightsquigarrow$ $b =$ LC of cols of A

Def The **span** of a list of vectors $\{v_1, \dots, v_n\}$ is the set of all linear combinations of $v_1, \dots, v_n$.

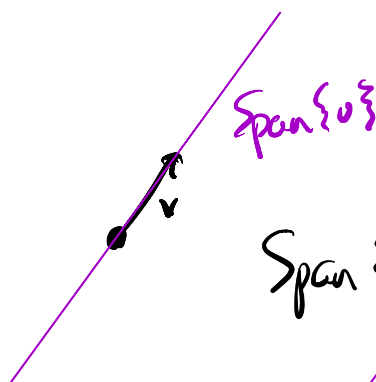
$$\mathrm{Span}\{v_1, v_2, v_3\} = \underbrace{\{}_{\text{"the set of all"}} \underbrace{x_1v_1 + x_2v_2 + x_3v_3}_{\text{"things of this form"}} \underbrace{:}_{\text{"such that"}} \underbrace{x_1, x_2, x_3 \in \mathbb{R}}_{\text{condition}} \}$$

set-builder notation

$$\boxed{\begin{array}{ccc} Ax=b \text{ consistent} & \Longleftrightarrow & b \text{ is in the span} \\ \text{(has a soln)} & & \text{of the columns of } A \end{array}}$$

Span is the smallest "linear space" (line, plane, ...) containing the vectors & 0.

Pictures:

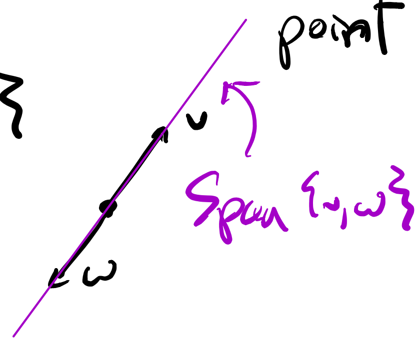


- $\mathrm{Span}\{v\} = \{xv : x \in \mathbb{R}\}$
  = line thru v & 0.

$\mathrm{Span}\{0\} = \{0\}$

point

- $\mathrm{Span}\{v, w\} = \{x_1v + x_2w : x_1, x_2 \in \mathbb{R}\}$



- $\mathrm{Span}\{\} = \{0\}$ (convention)

Q: How many solutions can $Ax=b$ have if Span{cols of A} is $\mathbb{R}^m$?

Def: A matrix has full row rank if it has a pivot in every <u>row</u>.

HW 2.8: back-substitution can't fail

Equivalently:
- $Ax=b$ is always consistent!
- Span{cols of A} $= \mathbb{R}^m$

Eg: $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ $Ax=b$ always consistent
Span{cols} $= \mathbb{R}^2$

$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $Ax = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ inconsistent
Span{cols} = xy-plane $\neq \mathbb{R}^3$

Parametric Vector Form

Can write solution sets using spans.

Eg: $\left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$\rightsquigarrow$ $\begin{aligned} x+5z &= 1 \\ y+2z &= -1 \end{aligned}$ $\xrightarrow[\text{form}]{\text{parametric}}$ $\begin{aligned} x &= -5z + 1 \\ y &= -2z - 1 \\ z &= z \end{aligned}$

← columnate

← include free variable

parametric vector form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ← "particular solution"

anything in $\mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$

Solutions: $\mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Procedure (Parametric Vector Form):

To find the parametric vector form of the solutions of $Ax=b$:

(1–3): Find the parametric form

(4): Organize the right side of the equations into columns & write as a vector equation

eg: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{x_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}}_{\mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}\right\}} + \begin{pmatrix} 1 \\ 0 \\ 9 \\ 0 \end{pmatrix}$ x_2, x_4: free variables

The solution set is a translate of a span:

$$\mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} 1 \\ 0 \\ 9 \\ 0 \end{pmatrix}$$ ← particular solution

Homogeneous Equations

Def: $Ax=b$ is homogeneous if $b=0$.

Q: How many solutions does $Ax=0$ have?

Important: $Ax=0$ is always consistent! It has the trivial solution $x=0$.

Eg:

$$\left[\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right] \xrightarrow{R_2 -= \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 0 & 3/2 & 3 & 0 \end{array}\right]$$

$$\cdots \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

(leave this out)

Inhomogeneous

$$\left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right]$$

RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{aligned} x + 5z &= 1 \\ y + 2z &= -1 \end{aligned}$$

$$\begin{aligned} x &= -5z + 1 \\ y &= -2z - 1 \\ z &= z \end{aligned}$$

Homogeneous

$$\left[\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\begin{aligned} x + 5z &= 0 \\ y + 2z &= 0 \end{aligned}$$

$$\begin{aligned} x &= -5z \\ y &= -2z \\ z &= z \end{aligned}$$

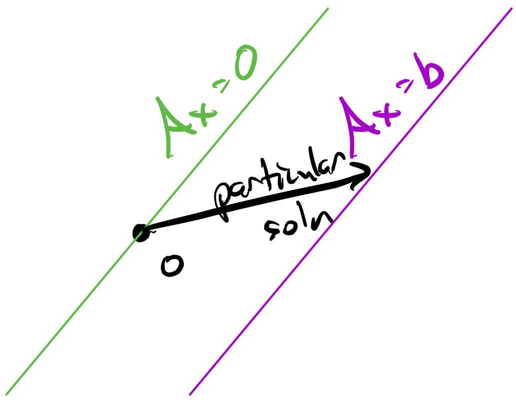
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

Solutions: $\mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\qquad \mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$

Upshots:

(1) The solution set of $Ax=0$ is a span

(2) The solution set of $Ax=b$ is empty or it is a translate of (parallel to) the solution set of $Ax=0$ (by a particular solution).



[Ax=b demo]

NB: Expressing a solution set as a translate of a span means writing it with parameters:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 9 \\ 0 \end{pmatrix}$$

x_2, x_4: parameters

So think

Spans $\longleftrightarrow$ Parametric Description