Odds & Ends

Def: The rank of a matrix is the number of pivots.
→ Need REF to compute

For an $m \times n$ matrix of rank r,
- full column rank: $r = n$
- full row rank: $r = m$
- invertible: $r = m = n$

Def: The transpose of a matrix is what you get by flipping over the diagonal:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

The rows of A are the columns of A^T, & vice-versa. A is $m \times n \iff A^T$ is $n \times m$.

Rules for transposes:

(1) $(AB)^T = \cancel{A^T B^T}$ $\quad B^T A^T$ ✓

$\quad (m \times n) \cdot (n \times p) \quad \cancel{(n \times m)(p \times n)} \quad (p \times n) \cdot (n \times m)$

(2) A invertible $\iff A^T$ invertible, in which case $(A^T)^{-1} = (A^{-1})^T$.

Check: $A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

$\implies (A^{-1})^T = (A^T)^{-1}$

(3) $(A^T)^T = A$

Def: A matrix S is symmetric if $S = S^T$

$\to$ NB S is square

Eg:
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Eg: A any matrix $\rightsquigarrow$ $S = A^TA$ is symmetric:

$$S^T = (A^TA)^T = A^TA^{TT} = A^TA = S$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 4 & 5 \end{bmatrix}$$

Subspaces

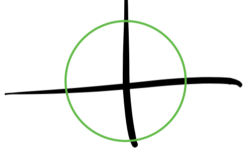
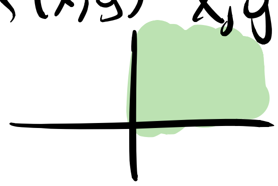
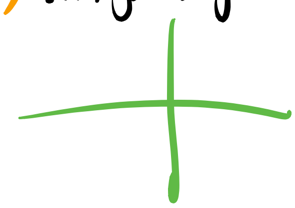
Any matrix $\rightsquigarrow$ two spans

- Span of columns (when $Ax=b$ consistent)
- Solutions of $Ax=0$ (what solutions look like)

To put these on the same footing, we'll formalize what we mean by "linear space thru 0".

Def: A subset of $\mathbb{R}^n$ is any collection of points

(1) $\{(x,y) : x^2+y^2=1\}$ (2) $\{(x,y) : x,y \geq 0\}$ (3) $\{(x,y) : xy=0\}$

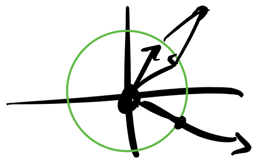
Def: A subspace of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ satisfying:

(1) If $u, v \in V$ then $u+v \in V$. [closed under addition]

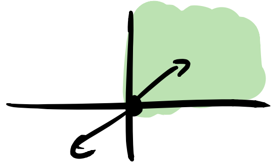
(2) If $u \in V$ an $c \in \mathbb{R}$ then $cu \in V$ [closed under scalar $\times$]

(3) $0 \in V$ [contains 0]

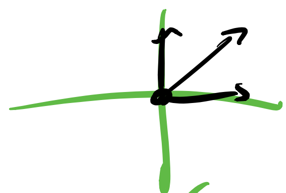
Q: Which properties do these fail?



~~(1)~~, ~~(2)~~, ~~(3)~~



(1) ✓ ~~(2)~~ (3) ✓



~~(1)~~ (2) ✓ (3) ✓

$$\boxed{\text{Any subspace is a span} \quad \& \quad \text{Any span is a subspace}}$$

Eg: $V = \{(x, y, x+y) : x, y \in \mathbb{R}\}$

(1) $(x_1, y_1, x_1+y_1), (x_2, y_2, x_2+y_2) \in V$

sum: $(x_1+x_2, y_1+y_2, x_1+y_1+x_2+y_2)$

$= (x_1+x_2, y_1+y_2, (x_1+x_2)+(y_1+y_2)) \in V$ ✓

(2) $(x, y, x+y) \in V \quad c \in \mathbb{R}$ ✓

$c(x, y, x+y) = (cx, cy, c(x+y)) = (cx, cy, cx+cy) \in V$

(3) $(0,0,0) = (0, 0, 0+0) \in V$ ✓

Conclusion: V is a subspace.

Check: $V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Eg: V = Solution set of $Ax=0$ $= \{x \in \mathbb{R}^n : Ax=0\}$

(1) $x, y \in V$ means $Ax=0$ $Ay=0$

$A(x+y) = Ax + Ay = 0+0 = 0 \Rightarrow x+y \in V$ ✓

(2) $x \in V$ $c \in \mathbb{R}$ know $Ax=0$

$A(cx) = cAx = c \cdot 0 = 0 \Rightarrow cx \in V$ ✓

(3) $A0 = 0 \Rightarrow 0 \in V$ ✓

Conclusion: V is a subspace.

Def: The null space of a matrix A is the solution set of $Ax=0$.

$$\mathrm{Nul}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

[row picture]

This is a subspace of $\mathbb{R}^n$ ($n = \#$ cols of A)

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightsquigarrow$ $\begin{matrix} x+2y=0 \\ z=0 \end{matrix}$ $\xrightarrow{\text{PF}}$ $\begin{matrix} x = -2y \\ y = y \\ z = 0 \end{matrix}$ $\xrightarrow{\text{PVF}}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Eg: $V = \mathrm{Span}\{v_1, v_2, v_3\}$

(1) $x_1v_1 + x_2v_2 + x_3v_3,\ y_1v_1 + y_2v_2 + y_3v_3 \in V$

sum: $x_1v_1 + x_2v_2 + x_3v_3 + y_1v_1 + y_2v_2 + y_3v_3$

$= (x_1+y_1)v_1 + (x_2+y_2)v_2 + (x_3+y_3)v_3 \in V$ ✓

(2) $x_1v_1 + x_2v_2 + x_3v_3 \in V$ $c \in \mathbb{R}$

$c(x_1v_1 + x_2v_2 + x_3v_3) = (cx_1)v_1 + (cx_2)v_2 + (cx_3)v_3 \in V$ ✓

(3) $0 = 0v_1 + 0v_2 + 0v_3 \in V$

Conclusion: V is a subspace.

This shows a span is a subspace.

Def: The **column space** of a matrix A is the span of the columns of A.

$$\mathrm{Col}(A) = \{ b \in \mathbb{R}^m : Ax = b \text{ is consistent} \}$$
$$= \{ Ax : x \in \mathbb{R}^n \}$$

[column picture]

This is a subspace of $\mathbb{R}^m$ ($m = \#\text{rows}$)

Eg:
$$\mathrm{Col}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

Don't say "span of a matrix" or "subspace of a matrix": there are ≥ 2 of them!