Last time

Def: The column space of a matrix is the span of the columns of the matrix.

$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m : Ax = b \text{ is consistent}\}$$
$$= \{Ax : x \in \mathbb{R}^n\}$$

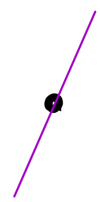
[column picture]

This is a subspace of $\mathbb{R}^m$ ($m = \#$ rows)

Eg: $\mathrm{Col}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

Q: Do row operations change $\mathrm{Col}(A)$?

Yes! $\mathrm{Col}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$



$\downarrow R_2 -= 2R_1$

$\mathrm{Col}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Def: The null space of a matrix A is the solution set of $Ax = 0$.

$$\mathrm{Nul}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

[row picture]

This is a subspace of $\mathbb{R}^n$ ($n = \#$ cols)

Q: Do row operations change $\mathrm{Nul}(A)$?

No! Solutions of $Ax = 0$ unchanged by row operations.

We know every subspace is a span. What is Nul(A) a span of?

Eg: $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \overset{RREF}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow x + 2y - z = 0 \overset{PE}{\rightsquigarrow} \begin{aligned} x &= -2y + z \\ y &= y \\ z &= z \end{aligned} \overset{PVF}{\rightsquigarrow} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Solution set: $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

Eg: If A has full column rank, $\text{Nul}(A) = \{0\} = \text{Span}\{\}$

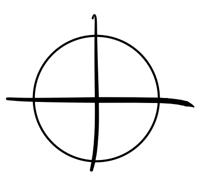
Procedure: To write Nul(A) as a span:
(1) Find the parametric vector form of the solutions of $Ax = 0$.
(2) $\text{Nul}(A) = \text{Span}\{\text{vectors attached to free variables}\}$

Think:

Nul(A): implicit form of a subspace
$x + y + z = 0$

$\overset{\text{elimination}}{\rightsquigarrow}$

Col(A)/Span: parameterized form of a subspace
$x_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\{(x,y) : x^2 + y^2 = 1\}$  $\{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\}$

The Four Fundamental Subspaces

Def: The row space of a matrix is the span of the rows. $\mathrm{Row}(A) = \mathrm{Col}(A^T)$. This is a subspace of $\mathbb{R}^n$. [row picture]

Eg: $\mathrm{Row}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

Q: Do row operations change Row(A)?

No! Say A has rows v_1, v_2, v_3 $\quad A = \begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix}$

- $R_1 \mathrel{+}= 2R_2$: $\quad \begin{pmatrix} - v_1 + 2v_2 - \\ - v_2 - \\ - v_3 - \end{pmatrix}$

  $\mathrm{Span}\{v_1 + 2v_2, v_2, v_3\} \overset{\checkmark}{=} \mathrm{Span}\{v_1, v_2, v_3\}$
- $R_1 \times= 2$ $\quad \begin{pmatrix} - 2v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix}$

  $\mathrm{Span}\{2v_1, v_2, v_3\} \overset{\checkmark}{=} \mathrm{Span}\{v_1, v_2, v_3\}$
- $R_1 \leftrightarrow R_2$ $\quad \begin{pmatrix} - v_2 - \\ - v_1 - \\ - v_3 - \end{pmatrix}$

  $\mathrm{Span}\{v_2, v_1, v_3\} \overset{\checkmark}{=} \mathrm{Span}\{v_1, v_2, v_3\}$

Def: The left null space of a matrix A is $\mathrm{Nul}(A^T)$. This is a subspace of $\mathbb{R}^m$. [column picture]

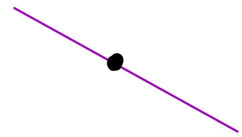
NB: $A^T x = 0 \iff (A^T x)^T = 0 \iff x^T A = 0$

$\Rightarrow$ left null space is $\{x \in \mathbb{R}^m : x^T A = 0\}$

$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

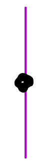
Q: Do row operations change the left null space?

Yes! $\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$ $\quad \mathrm{Nul}\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix}-2\\ 1\end{pmatrix}\right\}$



$\Big\downarrow \{R_2 -= 2R_1$

$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$ $\quad \mathrm{Nul}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix}0\\ 1\end{pmatrix}\right\}$



Q: Relationship to pictures of col space?

Name	Subspace of	Row/col picture	Changed by row ops?
$\mathrm{Col}(A)$	$\mathbb{R}^m$	col	yes
$\mathrm{Nul}(A)$	$\mathbb{R}^n$	row	no
$\mathrm{Row}(A)$	$\mathbb{R}^n$	row	no
$\mathrm{Nul}(A^T)$	$\mathbb{R}^m$	col	yes

Important: To do computations on subspaces, step 0 is write it as $\mathrm{Col}(A)/\mathrm{Span}\{\cdots\}$ (parameterized form) or $\mathrm{Nul}(A)$ (implicit form)

Eg: $V = \{(x,y,z) : x+y+2z=0\}$
$= \mathrm{Nul}\begin{pmatrix}1 & 1 & 2\end{pmatrix}$