Linear Independence

HW #3.12:

$\mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ is the plane $-13b_1 - 5b_2 + b_3 = 0$

But if you solve $Ax=0$ in PVF, 3 parameters $\Rightarrow$ null space is "space" (3-dimensional). What's the difference?

They're coplanar!

$$\begin{pmatrix}-1\\5\\12\end{pmatrix} = \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

So want a notion of "some vector is a linear combo of the others." Which one?

$$\begin{pmatrix}2\\-4\\6\end{pmatrix} = \frac{2}{5}\begin{pmatrix}-1\\5\\12\end{pmatrix} + \frac{6}{5}\begin{pmatrix}2\\-5\\1\end{pmatrix} \qquad \begin{pmatrix}2\\-5\\1\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}-1\\5\\12\end{pmatrix} + \frac{5}{6}\begin{pmatrix}2\\-4\\6\end{pmatrix}$$

Def: A set of vectors $\{v_1, \dots, v_n\}$ is **linearly dependent (LD)** if the homogeneous vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$$

has a nontrivial solution. Such a solution is a **linear dependence relation.**

Eg: $5\begin{pmatrix}2\\-4\\6\end{pmatrix} - 6\begin{pmatrix}2\\-5\\1\end{pmatrix} - \begin{pmatrix}-1\\5\\12\end{pmatrix} = 0$

is a linear dependence relation.

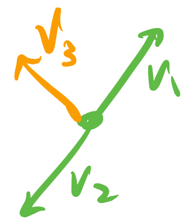
Given a linear dependence relation $x_1v_1 + x_2v_2 + x_3v_3 = 0$

nontrivial $\implies$ some $x_i \neq 0$. Suppose $x_1 \neq 0$. Then

$$v_1 = \frac{1}{x_1}(x_2 v_2 + x_3 v_3) \implies v_1 \in \operatorname{Span}\{v_2, v_3\}$$

LD: some vector is in the span of the others.

NB: this doesn't say every vector is in the span of the others!



$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Linear dependence relation: $v_1 + v_2 + 0\,v_3 = 0$

$$\rightsquigarrow v_1 = -v_2 \in \operatorname{Span}\{v_2, v_3\}$$

Def: A set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent (LI) if it's not linearly dependent, i.e.,

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only the trivial solution.

LI: no vector is in the span of the others.

A set of vectors is LI if its span is as big as you expect it to be.

i.e. 1 vector $\rightsquigarrow$ line
2 vectors $\rightsquigarrow$ plane
3 vectors $\rightsquigarrow$ space etc.

Eg: Are

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\10\end{pmatrix} \quad \text{LI?}$$

Need to solve $\quad x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} + x_3\begin{pmatrix}7\\8\\10\end{pmatrix} = 0$

$$\rightsquigarrow \begin{pmatrix}1&4&7\\2&5&8\\3&6&10\end{pmatrix} x = 0 \quad \overset{\text{REF}}{\rightsquigarrow} \begin{pmatrix}1&4&7\\0&1&2\\0&0&1\end{pmatrix}$$

No free vars $\Rightarrow$ no nontrivial solns $\Rightarrow$ LI.

NB: If $Ax=0$ has a nontrivial solution $(x_1, \ldots, x_n)$
$\Rightarrow x_1 v_1 + \cdots + x_n v_n = 0$ is a linear dependence relation.

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

Matrix interpretation:

A has LI columns
$\iff Ax=0$ has only the trivial soln
$\iff \mathrm{Nul}(A) = \{0\}$
$\iff A$ has full column rank.

Language: LI/LD are adjectives that describe a set of vectors.

Bad: "A is LI" "v_1 is LD on v_2 & v_3"

Good: "A has LI columns" "$\{v_1, v_2, v_3\}$ is LD"

Eg:
- $\{v\}$ is LI $\iff v \neq 0$
- Any set containing 0 is LD: if $v_1 = 0 \Rightarrow$

$1v_1 + 0v_2 + \cdots + 0v_n = 0$ LDR

- $\{v_1, w\}$ is LD $\iff v_1 \in \mathrm{Span}\{v_2\}$ or $v_2 \in \mathrm{Span}\{v_1\}$
  $\iff v_1, v_2$ collinear
  $\iff \mathrm{Span}\{v_1, v_2\}$ is a line (or $\{0\}$)
- Similarly, $\{v_1, v_2, v_3\}$ LD $\iff$ coplanar

Basis

Like: writing a subspace V as $\mathrm{Span}\{v_1, v_2, v_3\}$:
any vector in V is $v = x_1v_1 + x_2v_2 + x_3v_3$

Don't like: this expression can be non-unique!
$0 = x_1v_1 + x_2v_2 + x_3v_3$ can have nontrivial solns.
This happens if you used too many vectors
(eg. 3 vectors to span a plane).

Def: A **basis** for a subspace V is a set of vectors $\{v_1, \ldots, v_d\}$ in V satisfying:
(1) $V = \mathrm{Span}\{v_1, \ldots, v_d\}$
(2) $\{v_1, \ldots, v_d\}$ is LI

"spans & is LI"

Spans: every vector in V can be written
$v = x_1v_1 + \cdots + x_dv_d$

LI: this expression is unique: only one choice of $(x_1, \ldots, x_d)$

Why? $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_d \\ | & & | \end{pmatrix}$ then $A\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = v$
has one solution. (Zero solns: $v \notin V$)

Basis $\{v_1, \dots, v_d\}$ makes V "look like" $\mathbb{R}^d$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \longleftrightarrow x_1 v_1 + \cdots + x_d v_d$$

Def: The unit coordinate vectors are the columns of I_n:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \dots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Eg: $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$

Spans: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ✓

LI: I_n has full column rank ✓

Bases of $\mathbb{R}^n$: Suppose the columns of A form a basis of $\mathbb{R}^n$. Let r = rank.

(1) Span: $\mathrm{Col}(A) = \mathbb{R}^n \iff$ full row rank $r = m$

(2) LI: $\mathrm{Nul}(A) = \{0\} \iff$ full col rank $r = n$

So $r = m = n$ so A is invertible.

$$\boxed{\text{Basis for } \mathbb{R}^n \iff \text{columns of an invertible matrix}}$$

Notes:

- $\mathbb{R}^n$ has many bases:

  eg. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$, ...

- Every basis of $\mathbb{R}^n$ has n vectors.

True for any subspace. $\rightsquigarrow$ dimension