Dimension

Def: A **basis** for a subspace V is a set of vectors $\{v_1, \dots, v_d\}$ in V satisfying:

(1) $V = \text{Span}\{v_1, \dots, v_d\}$ "Spans & LI"

(2) $\{v_1, \dots, v_d\}$ is LI

$\rightarrow$ minimal spanning set for V.

Last time: basis for $\mathbb{R}^n \longleftrightarrow$ cols of an invertible $n \times n$ matrix

$\rightarrow$ Any basis has n vectors.

Facts:

(1) Every subspace has a basis.

(2) Every basis for V has the same # vectors in it.

Def: The **dimension** of a subspace V is

$\dim(V) = $ # vectors in any basis for V.

Eg:

- $\dim(\mathbb{R}^n) = n$ ✓
- If $\{v_1, \dots, v_d\}$ is LI $\Rightarrow \dim \text{Span}\{v_1, \dots, v_d\} = d$
  b/c $\{v_1, \dots, v_d\}$ is a basis for its span.
- If $v \neq 0 \Rightarrow \{v\}$ is LI $\Rightarrow \dim \text{Span}\{v\} = 1$
  $\rightarrow$ a line is 1-dimensional

- If v, w not collinear $\Rightarrow \{v, w\}$ is LI
  $\Rightarrow \dim \text{Span}\{v, w\} = 2$
  $\rightarrow$ a plane is 2-dimensional
- Q: What is $\dim\{0\}$?
  $\dim\{0\} = 0$: basis is $\{\}$ $\quad \{0\} = \text{Span}\{\}$
  $\rightarrow$ points are 0-dimensional

Bases for the Four Fundamental Subspaces

A: $m \times n$ matrix of rank r (r pivots)

eg. $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix}$ $\quad 2 \times 3$, $r = 1$

Nul(A)

Thm: The vectors in the parametric vector form of the solns of $Ax = 0$ form a basis for Nul(A).

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow x + 2y - z = 0$$

$$\xrightarrow{\text{PF}} \begin{aligned} x &= -2y + z \\ y &= y \\ z &= z \end{aligned} \quad \xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

check

Spans: every solution of $Ax = 0$ has this form ✓

LI: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} y &= 0 \\ z &= 0 \end{aligned}$ ✓

NB: This explains why
"1 free variable $\rightsquigarrow$ soln set is a line"
"2 free variables $\rightsquigarrow$ soln set is a plane" ...

dim Nul(A) = # free variables = # cols w/no pivot
= n − r

Col(A)

Thm: The pivot columns of A form a basis of Col(A).

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{basis: } \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

pivot column in A, not in RREF
(A & RREF have different Col spc)

dim Col(A) = # pivots = r = rank(A)

why?

$$A \xrightarrow{\text{RREF}} R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot cols
free cols

(col 3) = 3(col 1) + 2(col 2)
(col 5) = 4(col 1) + 6(col 2) − (col 4)

$$\Longleftrightarrow R\begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} = 0 \qquad R\begin{pmatrix} 4 \\ 6 \\ 0 \\ -1 \\ -1 \end{pmatrix} = 0$$

$$\overset{\text{Nul}(R) = \text{Nul}(A)}{\Longleftrightarrow} A\begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} = 0 \qquad A\begin{pmatrix} 4 \\ 6 \\ 0 \\ -1 \\ -1 \end{pmatrix} = 0$$

$\Longleftrightarrow$ (col 3) = 3(col 1) + 2(col 2)
(col 5) = 4(col 1) + 6(col 2) − (col 4) } cols of A

!!! R & A have same linear dependence relations among their columns.

check basis

Span: col 3 $\in$ Span {pivot cols}

col 5 $\in$ Span {pivot cols}

$\Rightarrow$ Col(A) = Span {pivot cols} ✓

LI: If a LC of pivot cols of A is 0

$\Longrightarrow$ a LC of pivot cols of R is 0

$$\begin{pmatrix}x_1\\x_2\\x_3\\0\end{pmatrix} = x_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 0 \Longrightarrow \begin{matrix}x_1=0\\x_2=0\\x_3=0\end{matrix}$$ ✓

Q: Find a basis for Span$\left\{\begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$

Systematic way: form the matrix with these columns:

$$\begin{pmatrix}1 & 2 & -1\\-2 & -4 & 2\end{pmatrix} \xrightarrow{RREF} \begin{pmatrix}1 & 2 & -1\\0 & 0 & 0\end{pmatrix}$$

& take pivot columns.

$\rightsquigarrow$ every span has a basis formed by deleting some vectors.

Row(A)

Thm: The nonzero rows of a (R)REF of A form a basis for Row(A).

$$\begin{bmatrix}1 & 2 & -1\\-2 & -4 & 2\end{bmatrix} \xrightarrow{REF} \begin{bmatrix}1 & 2 & -1\\0 & 0 & 0\end{bmatrix} \qquad \text{basis: } \left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$$

dim Row(A) = # nonzero rows in REF

= # pivots = rank(A)

why? Row(A) is unchanged by row ops ⇝ assume A is in REF.

Span: deleting zero rows doesn't change the span ✓

LI:
$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
✓

$$x_1\begin{pmatrix}1\\0\\3\\0\\4\end{pmatrix} + x_2\begin{pmatrix}0\\1\\2\\0\\6\end{pmatrix} + x_3\begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix} = 0 \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}$$

forward-subst.

Nul(A^T):

Thm: Suppose $EA = U$, for U a REF of A. Then the last $\underbrace{m-r}_{\#\text{ zero rows of } U}$ rows of E form a basis for Nul(A^T).

(E: product of elementary matrices)

This allows you to compute bases for all 4 subspaces by doing Gauss-Jordan once.
(as opposed to row reducing A^T).

Trick: augment A by the $m \times m$ identity matrix:

$$\underbrace{\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ -2 & -4 & 2 & 0 & 1 \end{array}\right]}_{A \,|\, I_m} \xrightarrow[R_2 \mathrel{+}= 2R_1]{\text{RREF}} \left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right] \quad \text{basis: } \left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}$$

$EA = U$ | $EI_m = E$

zero row of U

A: $m \times n$ of rank r

Name	row/col	dimension	basis
Nul(A)	row	$n-r$ ← # free variables	vectors in PVF
Col(A)	col	r ← # pivot cols	pivot columns
Row(A)	row	r ← # nonzero rows in REF	nonzero rows in a REF
Nul(A^T)	col	$m-r$ ← # zero rows in REF	last $m-r$ cols of E

Consequences

- rank = "row rank" = "column rank"

  $r = \dim \text{Row}(A) = \dim \text{Col}(A)$

  NB: A, A^T have different pivots

  → but same number of pivots

- Rank-Nullity:

  $\dim \text{Nul}(A) + \dim \text{Col}(A) = n = \#\text{cols}$

  $\dim \text{Nul}(A^T) + \dim \text{Row}(A) = m = \#\text{rows}$