Basis Thm: Let V be a subspace of dimension d and let $v_1, \ldots, v_d \in V$. Then $\{v_1, \ldots, v_d\}$ is a basis of V if
(1) $\{v_1, \ldots, v_d\}$ is LI OR (2) $V = \mathrm{Span}\{v_1, \ldots, v_d\}$

If you have the correct # vectors, don't need to check both!

Eg: two vectors v, w form a basis for a plane if
(1) not collinear (LI) or (2) span the plane.

See: Week 5 supplement

Orthogonal Complements

Recall: v is orthogonal to w, $v \perp w$, if $v \cdot w = 0$

NB: $v \cdot w = v^T w$ b/c $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$

Eg: Find all vectors $\perp$ $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Want to solve $v \cdot x = 0$, ie. $v^T x = 0$, ie. $\mathrm{Nul}(v^T)$.

$v^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \rightsquigarrow x + y + z = 0$

$\xrightarrow{\text{PF}}$ $x = -y - z$, $y = y$, $z = z$ $\xrightarrow{\text{PVF}}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

check: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0$ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0$ ✓

Eg: Find all vectors $\perp$ to $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ & $v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$.
Now we're trying to solve $v_1^T x = 0$ & $v_2^T x = 0$.

ie $\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix} x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \end{pmatrix} = 0$

So want to compute $\text{Nul}\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{matrix} x + y = 0 \\ z = 0 \end{matrix}$

$\xrightarrow{\text{PE}} \begin{matrix} x = -y \\ y = y \\ z = 0 \end{matrix} \xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

check: $\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\0\end{pmatrix} = 0 \quad \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\0\end{pmatrix} = 0$ ✓

NB: if $x \perp v_1$ & $x \perp v_2$ then
$(a v_1 + b v_2) \cdot x = a \cancel{v_1 \cdot x}^{0} + b \cancel{v_2 \cdot x}^{0} = 0$
so $x \perp v_1$ & $x \perp v_2 \iff x \perp \text{Span}\{v_1, v_2\}$

$$\boxed{\left\{ \begin{matrix} \text{All vectors } \perp \\ \text{Span}\{v_1, \dots, v_n\} \end{matrix} \right\} = \text{Nul}\begin{pmatrix} - v_1^T - \\ \vdots \\ - v_n^T - \end{pmatrix}}$$

Def: Two subspaces V, W of $\mathbb{R}^n$ are **orthogonal** if $v \cdot w = 0$ for all $v \in V$ & all $w \in W$.
The **orthogonal complement** of V is

$$V^\perp = \{ w \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } v \in V \}$$

"V perp" $\quad V^\perp \neq A^T$

This is also a subspace of $\mathbb{R}^n$.

Eg: $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$

$\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}$ (before)

NB: $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ & $\mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$ are orthogonal

but $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}^\perp = \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\} \neq \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$

Facts: let V be a subspace of $\mathbb{R}^n$.

(1) $(V^\perp)^\perp = V$ $\qquad$ (2) $\dim V + \dim V^\perp = n$

Orthogonality of the Four Subspaces

Say A has columns $v_1, \dots, v_n$.

$$\mathrm{Col}(A)^\perp = \mathrm{Span}\{v_1,\dots,v_n\}^\perp = \mathrm{Nul}\begin{pmatrix}- & v_1^T & -\\ & \vdots & \\ - & v_n^T & -\end{pmatrix} = \mathrm{Nul}(A^T)$$

Take $(\cdot)^\perp$ of both sides: $\mathrm{Nul}(A^T)^\perp = \mathrm{Col}(A)$

Take transposes: $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$ $\quad \mathrm{Nul}(A)^\perp = \mathrm{Row}(A)$

Orthogonality of the Four Subspaces

$\mathrm{Col}(A)$ & $\mathrm{Nul}(A^T)$ are orthogonal complements

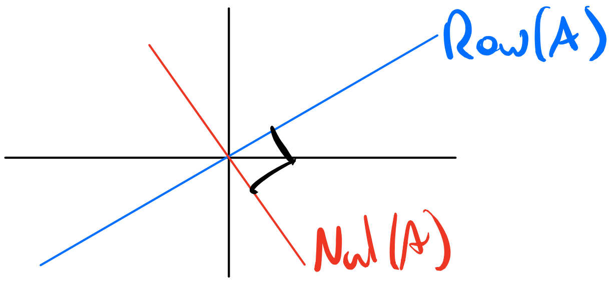
$\mathrm{Nul}(A)$ & $\mathrm{Row}(A)$ are orthogonal complements

Eg: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

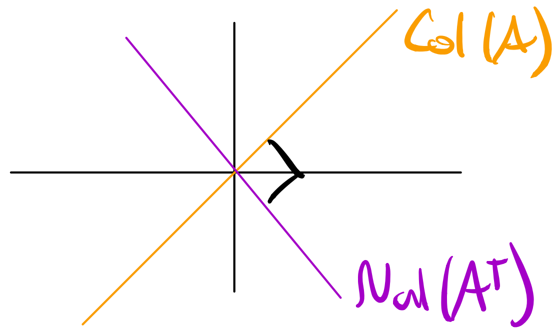
$$\text{Nul}(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

$$\text{Nul}(A^T) = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Row Picture



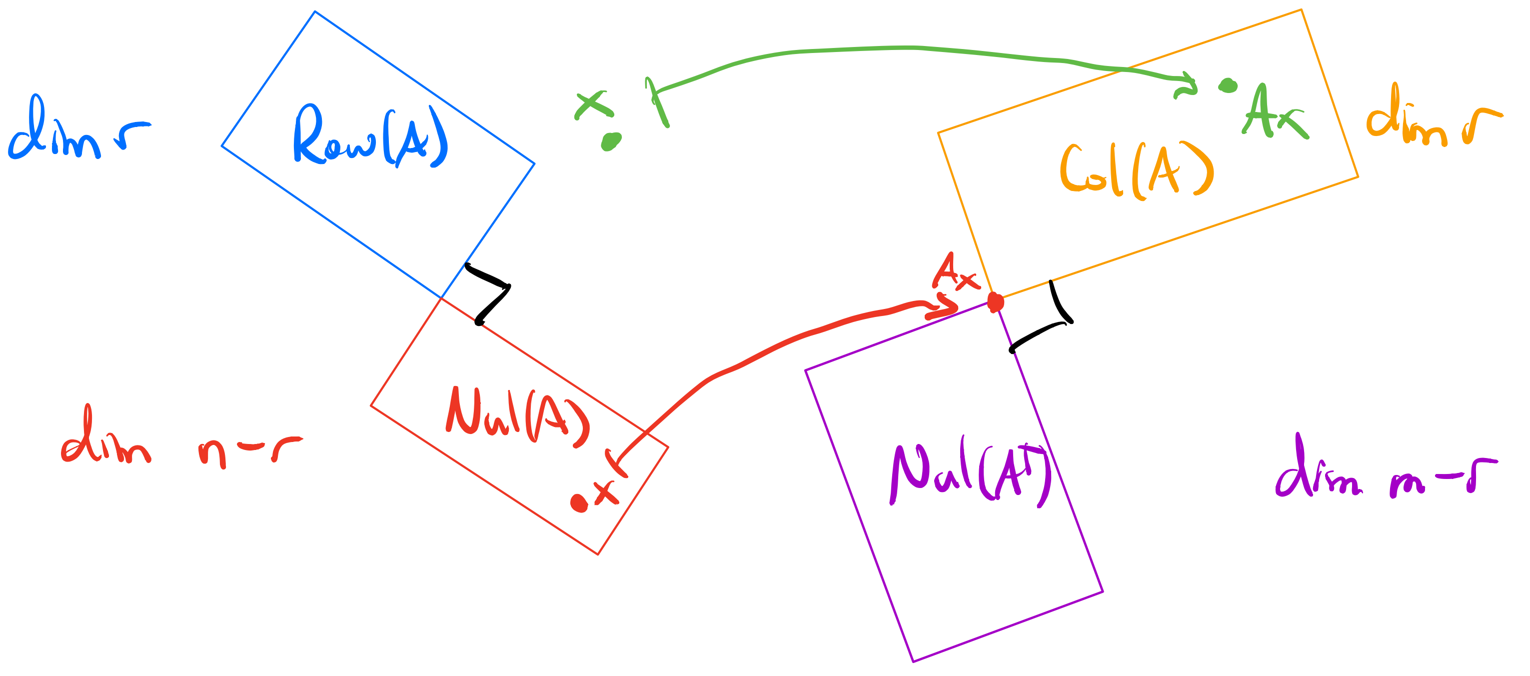
Column Picture



BIG PICTURE

Row Picture: $\mathbb{R}^n$

Column Picture: $\mathbb{R}^m$



Now you can do computations with orthogonal complements!

Eg: $V = \{(x,y,z) : x+2y=z,\ x+y+z=0\}$
Find a basis for $V^\perp$.

$$V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow V^\perp = \mathrm{Row}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$V^\perp = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

not collinear $\Rightarrow$ basis ✓

Recall: the (i,j) entry of A^TA is $(i^{th}\text{ col}) \cdot (j^{th}\text{ col})$

$$\underbrace{\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ — v_3^T — \end{pmatrix}}_{A^T} \underbrace{\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}}_{A} = \begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & v_1\cdot v_3 \\ v_2\cdot v_1 & v_2\cdot v_2 & v_2\cdot v_3 \\ v_3\cdot v_1 & v_3\cdot v_2 & v_3\cdot v_3 \end{pmatrix}$$

symmetric

Important Fact: $\boxed{\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)}$ HW4.13

Proof: Always $\mathrm{Nul}(A)$ is contained in $\mathrm{Nul}(BA)$

So want $\mathrm{Nul}(A^TA)$ to be contained in $\mathrm{Nul}(A)$.

$x \in \mathrm{Nul}(A^TA)$ means $A^TAx = 0$

Ax: in $\mathrm{Col}(A)$ in $\mathrm{Nul}(A^T)$ b/c $A^T(Ax) = 0$

so $Ax \perp Ax \Rightarrow Ax = 0 \Rightarrow x \in \mathrm{Nul}\ A$ ✓

Implicit Equations

Null space: implicit form for V $\xrightarrow{\text{PVF}}$ Span/col space: parametric form for V

orthogonal complements

Why? easy to check if $x \in V$: is $Ax = 0$?

Why? easy to produce all $x \in V$: $x = a_1 v_1 + \cdots + a_n v_n$

Eg: Find implicit equations $V = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

Trick: $V^{\perp} = \mathrm{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix} \overset{\text{PVF}}{=} \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\} = \mathrm{Col}\begin{pmatrix}-1 & -1\\ 1 & 0\\ 0 & 1\end{pmatrix}$

$$\Rightarrow V = \mathrm{Col}\begin{pmatrix}-1 & -1\\ 1 & 0\\ 0 & 1\end{pmatrix}^{\perp} = \mathrm{Nul}\begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$$

$$\Rightarrow V = \{(x, y, z) : -x + y = 0,\ -x + z = 0\}$$