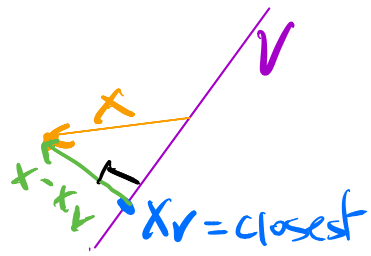
Orthogonal Projections

Question: If V is a subspace & x is a vector, what's the closest vector to x in V?

Closest: the vector $x_V \in V$ minimizing $\|x - x_V\|$

Observe: x_V is the closest vector

$\iff x - x_V \perp V \iff x - x_V \in V^\perp$.



Def: The orthogonal projection of x onto V is the closest vector x_V to x in V. It's characterized by:

$$x_{V^\perp} = x - x_V \in V^\perp.$$

The orthogonal decomposition of x relative to V is

$$x = \underbrace{x_V}_{\text{closest vector in } V} + \underbrace{x_{V^\perp}}_{\text{closest vector in } V^\perp}.$$

NB: $x_{V^\perp}$ is the projection of x onto $V^\perp$:

$$x - x_{V^\perp} = x_V \in V = (V^\perp)^\perp$$

How to compute?

Step 0: suppose $V = \mathrm{Col}(A)$.

Want $x_V \in V = \mathrm{Col}(A) \Rightarrow x_V = A\hat{x}$ for $\hat{x} \in \mathbb{R}^n$ "a linear combo of cols of A"

$V^\perp = \mathrm{Nul}(A^T)$ want $x - x_V \in V^\perp$

$$\Rightarrow A^T(x - x_V) = 0 \Rightarrow A^T x = A^T x_V = A^T A\hat{x}$$

Solve $A^TA\hat{x}=A^Tx \rightsquigarrow x_V = A\hat{x}$

Eg: Project $\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto $\mathrm{Col}\begin{pmatrix}1&1\\1&1\\1&0\end{pmatrix}$.

$$A^TA = \begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\\1&0\end{pmatrix} = \begin{pmatrix}3&2\\2&2\end{pmatrix}$$

$$A^Tx = \begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

Solve $A^TA\hat{x}=A^Tx$:

$$\left(\begin{array}{cc|c}3&2&1\\2&2&1\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c}1&0&0\\0&1&1/2\end{array}\right) \Rightarrow \hat{x} = \begin{pmatrix}0\\1/2\end{pmatrix}$$

$$\Rightarrow x_V = A\hat{x} = \begin{pmatrix}1&1\\1&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$$

$$x_{V^\perp} = x - x_V = \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}$$

orthogonal decomposition: $\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\0\end{pmatrix} + \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}$

Distance of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ from $\mathrm{Col}(A)$: $\|x_{V^\perp}\| = \left\|\begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}\right\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$.

Def: The normal equation of $Ax=b$ is $A^TAx=A^Tb$

Procedure: To find the orthogonal decomposition of x relative to $\mathrm{Col}(A)=V$:

(1) Solve the normal equation $A^TA\hat{x}=A^Tx$ for $\hat{x}$

(2) $x_V = A\hat{x}$ $\quad x_{V^\perp} = x - x_V$

Q: How to project onto $\mathrm{Nul}(A)$?

Eg: Project $\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto $\mathrm{Nul}\begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}$.

$$V^\perp = \mathrm{Row}\begin{pmatrix}1&1&1\\1&1&0\end{pmatrix} = \mathrm{Col}\begin{pmatrix}1&1\\1&1\\1&0\end{pmatrix}$$

$$\Rightarrow x_{V^\perp} = \begin{pmatrix}1/2\\1/2\\0\end{pmatrix} \Rightarrow x_V = x - x_{V^\perp} = \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}.$$

Moral: Decide first if it's easier to project onto V or $V^\perp$, then compute the easier one!

What if $V = \mathrm{Span}\{v\}$ is a line? $V = \mathrm{Col}(A)$, $A = v$

$$A^TA\hat{x} = A^Tx \iff v^Tv\hat{x} = v^Tx \iff (v\cdot v)\hat{x} = v\cdot x$$

$$\Rightarrow \hat{x} = \frac{v\cdot x}{v\cdot v} \Rightarrow x_V = A\hat{x} = \frac{v\cdot x}{v\cdot v}\,v$$

Projection onto a Line: If $V = \mathrm{Span}\{v\}$, $v \neq 0$, then

$$\boxed{x_V = \frac{v\cdot x}{v\cdot v}\,v}$$

Eg: Project $\begin{pmatrix}1\\0\end{pmatrix}$ onto $\mathrm{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

$$x_V = \frac{\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

Properties of Projections:

(1) $x_V = x \iff x \in V$

(2) $x_V = 0 \iff x \in V^\perp$

orthogonal decomposition

$x = x_V + 0$

$x = 0 + x_{V^\perp}$

Project Matrices

HW#6.17

Fact: A has full column rank $\iff A^TA$ is invertible

Suppose A has full col rank, $V = \mathrm{Col}(A)$.

cols of A are a basis for V

To compute x_V:

$$A^TA\hat{x} = A^Tx \iff \hat{x} = (A^TA)^{-1}A^Tx$$

$$\implies x_V = A\hat{x} = A(A^TA)^{-1}A^Tx$$

If $P = A(A^TA)^{-1}A^T$ then $x_V = Px$.

Eg: $V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow A^TA = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \longrightarrow (A^TA)^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$

$$P = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(don't compute by hand)

Check: $x_V = Px$, $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$P\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \checkmark$$

Def: Let V be a subspace. The projection matrix onto V is the matrix P_V such that

$$P_V x = x_V \text{ for all } x.$$

Procedure: To compute P_V:

(1) Find a basis for V.

(2) Let A = the matrix with those vectors

A has full col rank, $V = \mathrm{Col}(A)$

(3) $P_V = A(A^TA)^{-1}A^T$

Eg: $V = \mathrm{Col}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{\text{use}} A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

What if $V = \mathrm{Span}\{v\}$ is a line? $V = \mathrm{Col}(A)$ $A = v$

$A^TA = v^Tv = v \cdot v$

$\Rightarrow P_V = v\left(\frac{1}{v \cdot v}\right)v^T = \frac{1}{v \cdot v} v v^T$ "outer product"

Eg: $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P_V = \frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Check: $x_V = P_V x$ $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P_V x = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

Projection Matrix for a Line

If $V = \mathrm{Span}\{v\}$ for $v \neq 0$ then

$$\boxed{P_V = \frac{1}{v \cdot v} v v^T}$$

Properties of Projection Matrices

HW #6.8

(1) $P^2 = P$ (2) $P^T = P$ (3) $\underline{\underline{I_n = P_V + P_{V^\perp}}}$

Moral: Decide first if it's easier to compute P_V or $P_{V^\perp}$, then compute the easier one!