Recall: $\{u_1, \dots, u_n\}$ $\quad Q = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$

- orthogonal: $u_i \cdot u_j = 0$ $i \neq j$, $u_i \neq 0$ or Q^TQ invertible & diagonal
- orthonormal: $u_i \cdot u_j = 0$ $i \neq j$, $u_i \cdot u_i = 1$ or $Q^TQ = I_n$

Properties: Say Q has orthonormal columns.

(1) $Q^TQ = I_n$

(2) Q has full column rank

(3) $(Qx) \cdot (Qy) = x \cdot y$

(4) $\|Qx\| = \|x\|$

(5) Let $V = Col(Q)$, P_V = projection matrix, then

$$\boxed{P_V = QQ^T}$$

Vector form:

(2) if $\{u_1, \dots, u_n\}$ is orthogonal / orthonormal then it's LI

(5)
$$x_V = P_V x = QQ^T x = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} - u_1^T - \\ \vdots \\ - u_n^T - \end{pmatrix} x$$
$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} u_1 \cdot x \\ \vdots \\ u_n \cdot x \end{pmatrix} = (u_1 \cdot x)u_1 + \cdots + (u_n \cdot x)u_n$$

Projection Formula: Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of a subspace V. Then

$$\boxed{x_V = (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2 + \cdots + (u_n \cdot x)u_n}$$

If $\{u_1, \dots, u_n\}$ is orthogonal then

$$\boxed{x_V = \frac{u_1 \cdot x}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot x}{u_2 \cdot u_2} u_2 + \cdots + \frac{u_n \cdot x}{u_n \cdot u_n} u_n}$$

$$\left(\frac{u_1}{\|u_1\|} \cdot x \right) \cdot \frac{u_1}{\|u_1\|}$$

orthogonal

Eg: $V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$x_V = \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Gram-Schmidt & QR

We like orthogonal sets.

How to make a set of vectors orthogonal?

Procedure (Gram-Schmidt)

Let $\{v_1, \dots, v_n\}$ be a basis of a subspace V.

(1) $u_1 = v_1$

(2) $u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1$

(3) $u_3 = v_3 \underbrace{- \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2}$ – (proj of v_3 onto $\operatorname{Span}\{u_1, u_2\}$)

⋮

(n) $u_n = v_n - \frac{u_1 \cdot v_n}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_n}{u_2 \cdot u_2} u_2 - \cdots - \frac{u_{n-1} \cdot v_n}{u_{n-1} \cdot u_{n-1}} u_{n-1}$

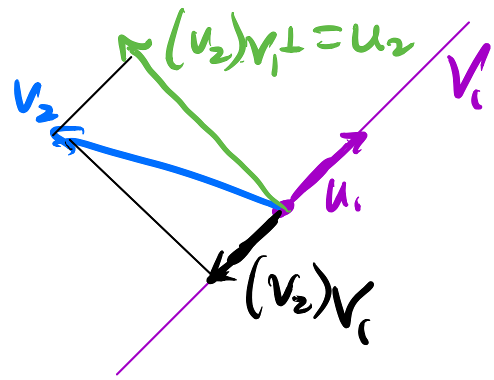
Then $\{u_1, \dots, u_n\}$ is an orthogonal basis of V, and $\operatorname{Span}\{u_1, \dots, u_i\} = \operatorname{Span}\{v_1, \dots, v_i\}$ for $1 \leq i \leq n$

Why? Projection formula!

$V_i = \mathrm{Span}\{u_1, \dots, u_i\}$

$$u_{i+1} = v_{i+1} - (v_{i+1})_{V_i} = (v_{i+1})_{V_i^\perp}$$

$$\Rightarrow u_{i+1} \perp u_1, \dots, u_i$$



Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

check: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0$ $\quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ $\quad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$

$u_1 \cdot u_2 = 0 \qquad u_1 \cdot u_3 = 0 \qquad u_2 \cdot u_3 = 0$ ✓

Q: What if $\{v_1, \dots, v_n\}$ is not LI?

If $v_{i+1} \in \mathrm{Span}\{v_1, \dots, v_i\} = \mathrm{Span}\{u_1, \dots, u_i\} = V_i$ then

$(v_{i+1})_{V_i} = v_{i+1}$ so $u_{i+1} = v_{i+1} - (v_{i+1})_{V_i} = 0$

So throw out v_i & continue.

$\rightsquigarrow$ still get an orthogonal basis.

Upshot: if $V = \mathrm{Col}(A)$, run G-S <u>without</u> first producing a basis.

Solve for v_i's in terms of u_i's:

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 + u_2 \\
v_3 &= \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 + u_3 \\
v_4 &= \frac{v_4 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v_4 \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{v_4 \cdot u_3}{u_3 \cdot u_3} u_3 + u_4
\end{aligned}
$$

repackage

Matrix Form:

$$
\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}
=
\begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix}
\begin{pmatrix}
1 & \frac{v_2 \cdot u_1}{u_1 \cdot u_1} & \frac{v_3 \cdot u_1}{u_1 \cdot u_1} & \frac{v_4 \cdot u_1}{u_1 \cdot u_1} \\
0 & 1 & \frac{v_3 \cdot u_2}{u_2 \cdot u_2} & \frac{v_4 \cdot u_2}{u_2 \cdot u_2} \\
0 & 0 & 1 & \frac{v_4 \cdot u_3}{u_3 \cdot u_3} \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

QR Decomposition

Let A be an $m \times n$ matrix with full column rank. Then

$$A = QR$$

where

- Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\mathrm{Col}(A)$.
- R is upper-Δ with positive diagonal entries

To compute Q & R: let $\{v_1, \ldots, v_n\}$ be the cols of A. Run Gram-Schmidt $\rightsquigarrow \{u_1, \ldots, u_n\}$. Then cancels

$$
\underbrace{\begin{pmatrix} | & | & | & | \\ \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} & \frac{u_4}{\|u_4\|} \\ | & | & | & | \end{pmatrix}}_{Q}
\underbrace{\begin{pmatrix}
\|u_1\| & \frac{v_2\cdot u_1}{u_1\cdot u_1}\|u_1\| & \frac{v_3\cdot u_1}{u_1\cdot u_1}\|u_1\| & \frac{v_4\cdot u_1}{u_1\cdot u_1}\|u_1\| \\
0 & \|u_2\| & \frac{v_3\cdot u_2}{u_2\cdot u_2}\|u_2\| & \frac{v_4\cdot u_2}{u_2\cdot u_2}\|u_2\| \\
0 & 0 & \|u_3\| & \frac{v_4\cdot u_3}{u_3\cdot u_3}\|u_3\| \\
0 & 0 & 0 & \|u_4\|
\end{pmatrix}}_{R}
$$

Eg:

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{-6}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{Q}\underbrace{\begin{pmatrix} \sqrt{2} & 1\sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{6} & -1\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{pmatrix}}_{R}$$

Analogous to LU decomposition:

$$A = \underbrace{L}_{\text{steps to get to echelon form}}\ \underbrace{U}_{\text{echelon form}} \qquad A = \underbrace{Q}_{\text{orthonormal basis}}\ \underbrace{R}_{\text{steps to get to o.n. basis}}$$

Why? Makes least squares faster! $A = QR$

Least-squares soln of $Ax = b$:

$$A^T A \hat{x} = A^T b$$

$$\cancel{R^T} \cancel{Q^T Q} R \hat{x} = \cancel{R^T} Q^T b$$

$$R\hat{x} = Q^T b$$

R is upper-Δ $\rightsquigarrow$ solve for $\hat{x}$ using back-substitution! $\sim \frac{1}{2} n^2$ flops if A is $n \times n$

NB: Can compute $A = QR$ in $\sim \frac{10}{3} n^3$ flops for $n \times n$ A.

not using this algorithm

Slower than $A = LU$: $\sim \frac{2}{3} n^3$ flops.